CPL Tokamak note(by Gil Refael and Isaac Kim)

Tokamak is a toroidal device using a magnetic field to confine plasma. First experimental research started back in the 1950s by a group Soviet scientists, hence the Russian name Tokamak. We are going to use a cylindrical coordinate system, so the unit vectors shall be represented as $\hat{z}$, $\hat{r}$, and $\hat{\theta}$. Direction of the magnetic field is $\hat{\theta}$ and our goal is to confine the particles inside the torus.

In this coordinate system, the force exerted on charged particle with mass m and charge q is the following.

$$\vec{F} = q\vec{v} \times \vec{B} = qB_0(v_r\hat{r} + v_\theta\hat{\theta} + v_z\hat{z}) \times \hat{\theta} \quad (1)$$
$$= qB_0v_r\hat{z} - qB_0v_z\hat{r}, \quad (2)$$

where B_0 is the strength of the magnetic field. First thing that we can immediately see from this equation is that there is no $\hat{\theta}$ component, which means that the angular momentum is conserved. Hence we have the following expression.

$$L = mr^2\dot{\theta} = Constant. \quad (3)$$

Next, we find that

$$F_z = qB_0v_r = qB_0\frac{dr}{dt} = m\frac{dv_z}{dt}. \quad (4)$$

Once we integrate this out by t, we get

$$v_z = v_{z,0} + \frac{qB_0}{m}(r - r_0), \quad (5)$$

where $v_{z,0}$ and r_0 are initial conditions. It seems that we need to find the expression for v_r as well, but it turns out that there is a simpler way. Remember that magnetic field alone cannot do any work: when we calculate $\vec{F} \cdot \vec{v}$ for magnetic field, direction of $\vec{F}$ is always orthogonal to direction of $\vec{v}$. Because of this relation, the total energy of the system is conserved and moreover the total energy is just identical to the kinetic energy of the particle. Hence we have the following nice formula.

$$E = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) \quad (6)$$
$$= \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + \frac{1}{2}m(v_{z,0} + \frac{qB_0}{m}(r - r_0))^2 \quad (7)$$

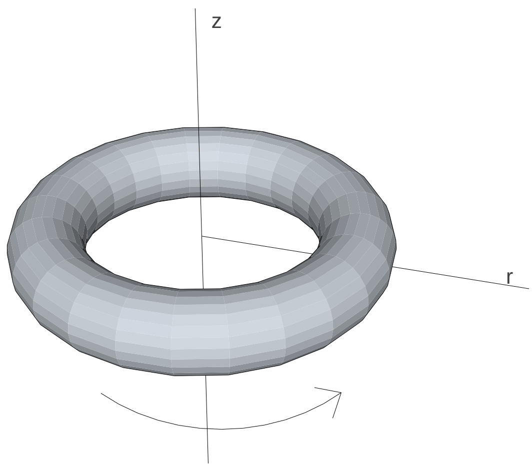

FIG. 1: Configuration of the Tokamak.

We can interpret this formula as a one dimensional system with mass m and *effective potential energy* $U = \frac{L^2}{2mr^2} + \frac{1}{2}m(v_{z,0} + \frac{qB_0}{m}(r - r_0))^2$. If you plot this as a function of r, it will have one stable point. Hence we can see that in the r direction, the charged particles will be confined. What about in the z direction? To have such confinement, time average of v_z should be zero. From Equation 5 we can see that when $r = r_0$, in other words if there is no movement in the radial direction, only way to make v_z 0 is to set $v_{z,0} = 0$. What if we have an oscillation in the radial direction? First note that the effective potential energy is not symmetric with respect to the value of r that minimizes it. This is in contrast to the case when we have a spring, hence having a symmetric potential energy like $U = \frac{1}{2}k(x - x_0)^2$. Due to this asymmetry, the particle will have a preferred direction at which it wants to stay longer. In this case, we have a steeper potential wall for small r, so the particle will spend more time for larger r, hence having a net positive drifting velocity in z direction. When the charge is negatively charged, the drifting velocity will be in $-\hat{z}$ direction.

Fortunately, when the plasma is neutral, the electric force between the positive charges and negative charges will be strong enough to cancel out the drifting effects. The real issue is that when the particles collide with each other, they do not always end up the having the desired reaction. They can simply recoil and this can potentially hinder the particle from following the stable trajectory that we calculated here.