Atmosphere vs. Astronomy: solutions

Problem 1

Differentiate Snell's law with respect to index of refraction n

$$n \sin\theta = \text{Const} \tag{1}$$
$$sin\theta + n\frac{d\theta}{dn} = 0$$
$$\frac{d\theta}{dn} = -\frac{1}{n}\tan\theta$$
$$d\theta = -dn \cdot \frac{\tan\theta}{n} \tag{2}$$

In this problem, we have $n \approx 1$, so $d\theta = -dn\tan\theta$.

Problem 2

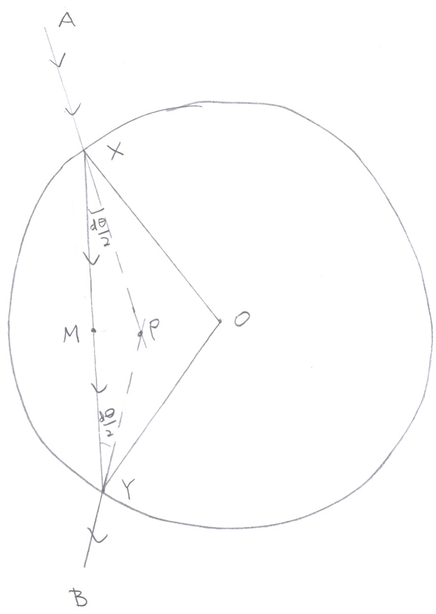

FIG. 1: Light ray bent due to thermal

See figure 1. The ray travels along $\overline{AX}$, $\overline{XY}$, then $\overline{YB}$. Let M be the midpoint of $\overline{XY}$, O the center of the circle, and P be the intersection of $\overline{AX}$ with $\overline{BY}$. The path must be symmetric across line $\overline{OM}$, thus the angle (to the normals) of the incoming and outgoing rays are equal, i.e., $\angle OXY = \angle OYX$. The angular deviations due to refractions are equal, $\angle YXP = \angle XYP = d\theta/2$. Using the result from problem 1, $d\theta/2 = -dn\tan\theta$. The total angular shift,

$$d\theta = \angle YPQ = \angle XYP + \angle YXP = -2dn\tan\theta. \tag{3}$$

A ray of light that passes through a diameter of the circle has no angular shift, but if it just grazes (tangent to) the circle, there is an unbounded shift: $\tan\pi/2 = \infty$. This angular shift is limited by a few different things:

Let L represent how far the ray penetrates into the circle, i.e., the distance MP for radius $\overline{OP}$ perpedicular to the ray (figure 2).

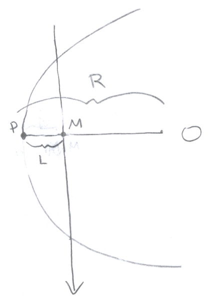

FIG. 2: L is distance from ray to boundary of thermal

A. The boundary of the thermal, dividing the regions with indices n and $n - dn$, is not sharp, the index of refraction varies over some length α. A ray with $L < \alpha$ will never see $n - dn$.

B. The aperture of the lens you use to observe the star: whether it is your eye, or a telescope, has diameter α. The ray of light that is displaced by a distance of α should have the same angle when it reaches the lens, otherwise the star will not produce an image on your retina. See figure 3.

C. The ray of light will pass through a finite number of thermals on the way to the ground, so for some small β, it is unlikely that $L < \alpha$ for any of the thermals it crosses.

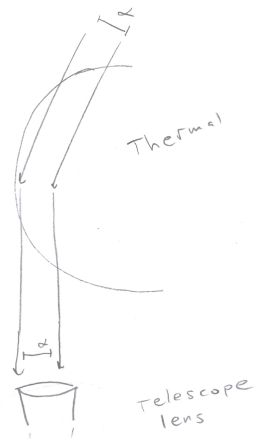

FIG. 3: Telescope doesn't produce an image unless $L > \alpha$

The consequence of (A.) and (B.) is the same: $L > \alpha$. Thus the maximum deviation occurs at $L = \max(\alpha_{\text{interface thickness}}, \alpha_{\text{lens diameter}})$

Let R be the radius of the spherical thermal, and $L/R = \epsilon$. Assuming $d\theta \ll 1$, $\sin\theta = \frac{R-L}{R} = 1 - \epsilon$. Then $\cos\theta = \sqrt{2\epsilon}$ and $\tan\theta = 1/\sqrt{2\epsilon}$.

Thus we get

$$d\theta = \sqrt{2R/L} \cdot dn \tag{4}$$

With $dT = 1\,\mathrm{C}$, $dn = 10^{-6}$. We take $2R = 50\,\mathrm{m}$ and take $\alpha = 10\,\mathrm{cm}$, which is a reasonable estimate for both a telescope lens and the thickness of the transition region.

$$\begin{aligned} d\theta &= \sqrt{2L/\alpha} \cdot dn & (5) \\ &= \sqrt{50\,\mathrm{m}/10\,\mathrm{cm}} \cdot 10^{-6} \cdot \left(\sqrt{\frac{10\,\mathrm{cm}}{\alpha}}\right) \\ &\approx 2 \cdot 10^{-5} \cdot \left(\sqrt{\frac{10\,\mathrm{cm}}{\alpha}}\right) & (6) \end{aligned}$$

Problem 3

The angular shift at an interface is $d\theta = dn \tan\theta$. As in the previous model, the angle between the incoming ray and the interface is bounded by about α/R, where α is the length scale of the telescope or the thickness of the hot/cold interace, and R is the characteristic length scale of the surfaces, i.e. how long you can go along one surface without running into another one.

First let us calculate the variance of the angular shift at a single interface. Take $\epsilon = \alpha/R$ and $\phi = \pi/2 - \theta$.

$$\begin{aligned} < d\theta^2 > &= dn^2 < \tan\theta^2 > \\ &= \int_0^{\pi/2-\epsilon} d\theta \tan^2\theta \\ &= \int_{\pi/2}^{\epsilon} -d\phi \tan(\pi/2-\phi)^2 \\ &\approx \int_{\epsilon}^{\infty} d\phi 1/\phi^2 \\ &= 1/\epsilon = R/\alpha \end{aligned} \tag{7}$$

The ray will undergo several angular shifts as it passes through several interfaces in the atmosphere. How many? Most of the mass of the atmosphere lies below 10 km, so the angular shift will be due to this lower part of the atmosphere. Thus we expect that a ray from space will cross about $N = 10\,\text{km}/50\,\text{m} = 200$ interfaces.

Variance is additive, so we get for the total angular deviation

$$< d\theta_{\text{total}}^2 >= N d\theta^2 = dn^2 \cdot NR/\alpha \tag{8}$$

With $dT = .1\,\text{C}$, $dn = 10^{-7}$. The standard deviation of the shift is

$$\Delta\theta = dn \cdot \sqrt{\frac{NR}{\alpha}} \tag{9}$$

$$\begin{aligned} &= 10^{-7}\sqrt{\frac{200 \cdot 50\,\text{m}}{10\,\text{cm}}} \cdot (\sqrt{\frac{10\,\text{cm}}{\alpha}})(\sqrt{\frac{N}{200}}) \\ &= 3 \cdot 10^{-5} \cdot (\sqrt{\frac{10\,\text{cm}}{\alpha}})(\sqrt{\frac{N}{200}}) \end{aligned} \tag{10}$$

The limits of perception

The resolution of the human eye, i.e. the minimal angle at which we can distinguish two parallel lines from one line, is $1/2$ arcminutes $= (1/2) \cdot (1/60)$ degrees $= (1/2) \cdot (1/60) \cdot (\pi/180)$ radians $= 10^{-4}$ radians

Thus with a telescope that performs $10\times$ or more magnification, the angular shift should have a noticible effect, according to the parameter estimates above. With the naked eye, we may have to adjust α because we no longer have a 10 cm lens, making the twinkling barely noticeable.

The angular size of Jupiter perceived from the earth is $2 \cdot 143 \cdot 10^6\,\text{m}/780 \cdot 10^9\,\text{m} = 4 \cdot 10^{-4}$ radians.