

Charged cylinder solutions

Problem 1

Let the ρ be the two dimensional resistivity of the solution.

$$\mathbf{J} = \frac{1}{\rho} \nabla V \quad (1)$$

The total current is the surface (line) integral around the cylinder

$$I = \int d\mathbf{n} \cdot \mathbf{J} = \frac{1}{\rho} \int d\mathbf{n} \cdot \mathbf{E} \quad (2)$$

Here, $d\mathbf{n}$ is the normal to the displacement vector $d\mathbf{l}$

Using Gauss' law, this becomes

$$I = \frac{1}{\rho} Q / \epsilon_0 \quad (3)$$

where Q is the excess charge at the cylinder that produces the field.

Now we want to solve for the potential V , so we need to find the field of point (or circle) in 2D. Gauss law gives

$$2\pi r \frac{\partial V}{\partial r} = Q / \epsilon_0 \quad (4)$$

$$\frac{\partial V}{\partial r} = \frac{Q / \epsilon_0}{2\pi r} \quad (5)$$

$$V = \frac{Q / \epsilon_0}{2\pi r} \ln r + \text{Const} \quad (6)$$

In the case of a cylinder of radius r_0 that is clamped to potential 0,

$$V = \frac{Q / \epsilon_0}{2\pi r} \ln \frac{r}{r_0} \quad (7)$$

Using (3) we get

$$V = \frac{I\rho}{2\pi r} \ln \frac{r}{r_0} \quad (8)$$

The metal plate has a constant potential. We can produce a constant potential at the plate by using an image charge: we put an oppositely charged cylinder on the opposite side of the plate. Then the potential at the metal plate is zero.

The potential at the cylinder is the sum of its own potential and the potential due to the image charge.

$$V = \frac{1}{2\pi} \rho I \ln \frac{r}{r_0} - \frac{1}{2\pi} \rho I \ln \frac{2L}{r_0} \quad (9)$$

$$= \frac{1}{2\pi} \rho I \ln \frac{r}{2L} \quad (10)$$

We can read off the resistance

$$R = \frac{1}{2\pi} \rho \ln \frac{2L}{r} \quad (11)$$

$$= \frac{1}{2\pi} \cdot \frac{.2 \Omega \text{m}}{z} \cdot \ln(200) \quad (12)$$

$$= \frac{.17 \Omega \text{m}}{z} \quad (13)$$

where z is the depth of the solution.

Note that we made an approximation: the charge distribution on the cylinder is not perfectly symmetrical if the cylinder has constant potential. However, this approximation is good because $r \ll L$.

Problem 2

$$\frac{dQ}{dt} = -I \quad (14)$$

$$= \int_S d\mathbf{n} \cdot \mathbf{J} \quad (15)$$

$$= -\frac{1}{\rho} \int_S d\mathbf{n} \cdot \nabla V \quad (16)$$

$$= -\frac{1}{\rho} \int_V dV \nabla^2 V \quad (17)$$

$$= -\frac{Q}{\epsilon_0 \rho} \quad (18)$$

Thus the charge decays exponentially with time constant $\tau = \epsilon_0 \rho$, regardless of the position or shape of the object.