

# Mars elevator solutions

## Problem 1

We will calculate the tension in the rope along its length. Notation:

- $R_m$  is radius of mars
- $R_a$  is radius of ballast asteroid
- $R$  is some radius along the length of rope  $R_m < R < R_a$
- $\rho$  is the volume density of the rope (and asteroid)
- $A$  is the cross-sectional area of the rope
- $M$  is the mass of mars

The tension at radius  $R$  must equal the difference between the gravitational force and the centrifugal force from the rope lying under  $R$ .

$$T_R = \int_{R_m}^R dR \rho A \left( -\Omega^2 R + \frac{GM}{R^2} \right) \quad (1)$$

$$= \rho A \left( -\frac{\Omega^2}{2} (R^2 - R_m^2) + \frac{GM}{R_m} - \frac{GM}{R} \right) \quad (2)$$

$$= \rho A \left( -\frac{\Omega^2}{2} (R - R_m)(R + R_m) + \frac{GM(R - R_m)}{R_m R} \right) \quad (3)$$

$$= \rho A (R - R_m) \left( \frac{\Omega^2}{2} (R + R_m) + \frac{GM}{R_m R} \right) \quad (4)$$

Note that  $\frac{GM}{R_a^2} \approx \Omega^2 R_a$ . Since  $R_a$  is only slightly larger than the radius of geosynchronous orbit. Also recall that  $R_a \approx 6R_m$ . It follows that the term on the right  $\frac{GM}{R_m R}$  is more than 6 times larger than the term on the left (and usually more). Therefore, we ignore the term on the left and get

$$T_R = \rho A (R - R_m) \frac{GM}{R_m R} \quad (5)$$

$$= \frac{GM M_{\text{rope}}}{R_m R} \quad (6)$$

Using the velocity of wave propagation in a rope

$$v = \sqrt{\frac{T}{\rho A}} = \sqrt{\frac{(R - R_m)GM}{R_m R}} \quad (7)$$

We want to find the propagation time

$$\int_{R_m}^{R_a} \frac{dR}{v} \quad (8)$$

First we will turn this into a dimensionless integral.

Let  $\alpha = \frac{R - R_m}{R_m}$ .

$$\frac{1}{v} = \sqrt{\frac{R_m R}{(R - R_m)GM}} \quad (9)$$

$$= \sqrt{\frac{R_m}{GM}} \sqrt{\frac{\alpha + 1}{\alpha}} \quad (10)$$

$dR = R_m d\alpha$  so

$$\int_{R_m}^{R_a} \frac{dR}{v} = \sqrt{\frac{R_m^3}{GM}} \int_0^{\frac{R_a - R_m}{R_m}} d\alpha \sqrt{\frac{\alpha + 1}{\alpha}} \quad (11)$$

Note that at the bottom of the rope, the tension is zero, so the propagation velocity at the bottom of the rope is zero. However, as we will see, the integral of  $1/v$  remains finite.

The integral  $\int d\alpha \sqrt{\frac{\alpha+1}{\alpha}}$  is analytically tractable, but we will be lazy and break this integral into two pieces where we have a decent approximation. We use  $\alpha \ll 1$  in the left-hand integral and  $\alpha \gg 1$  on the right:

$$\int_0^{\alpha_{\max}} d\alpha \sqrt{\frac{\alpha + 1}{\alpha}} = \int_0^1 d\alpha \sqrt{\frac{\alpha + 1}{\alpha}} + \int_0^{\alpha_{\max}} d\alpha \sqrt{\frac{\alpha + 1}{\alpha}} \quad (12)$$

$$\approx \int_0^1 d\alpha \frac{1}{\sqrt{\alpha}} + \int_1^{\alpha_{\max}} d\alpha 1 \quad (13)$$

$$= 2 + (\alpha_{\max} - 1) \quad (14)$$

$$= 1 + \alpha_{\max} \quad (15)$$

$$= \frac{R_a}{R_m} \quad (16)$$

Note that the divergence of  $1/v$  only contributes approximately 1 to the integral. Substituting this into (11) we get

$$\text{Time} = \sqrt{\frac{R_m R_a^2}{GM}} \quad (17)$$

$$\approx \sqrt{\frac{3400 \text{ km} \cdot (21000 \text{ km})^2}{6.67 \cdot 10^{-11} \frac{\text{m}^3}{\text{kg} \cdot \text{s}^2} \cdot 6.42 \cdot 10^{23} \text{ kg}}} \quad (18)$$

$$= 5927 \text{ s} = 1.6 \text{ hours} \quad (19)$$

## Problem 2

The small asteroid has diameter 500 m, compared to the ballast asteroid's 5 km. Therefore the small asteroid weighs 1/1000 as much. Assuming an inelastic collision between the small asteroid and the ballast, the ballast is moving at a velocity of  $5 \frac{\text{km}}{\text{s}} / 1000 = 5 \frac{\text{m}}{\text{s}}$  after the collision.

Recall from the notes on wave propagation that

$$\tan \theta = \frac{v}{u} \quad (20)$$

where  $u$  is the propagation velocity in the line  $\sqrt{\frac{T}{\lambda}}$  and  $v$  is the velocity that we are pulling the rope at its end.

One tricky issue is that  $v$  changes as the wave propagates along the rope—this is the speed that the tip of the wave is being pulled up. We must make the approximation that the wave on the rope is not reflected, and only contains forward-moving components. In reality, there are small reflections since the impedance of the rope changes with  $R$ . We can then make the approximation that the energy flux  $\frac{1}{2} \rho u v^2$  is constant. Thus  $v \propto u^{-1/2}$ .<sup>1</sup>

Let  $T_{\text{top}}$  and  $T_{\text{base}}$  be the tension at the top and base of the rope, respectively.

$$T_{\text{top}} = \rho A (R_a - R_m) \frac{GM}{R_m R_a} = 4.1 \cdot 10^{12} N \quad (21)$$

$$T_{\text{base}} = GM m_{\text{cube}} / R_m^2 = 1.18 \cdot 10^9 N \quad (22)$$

$$v_{\text{base}}/v_{\text{top}} = \sqrt{u_{\text{top}}/u_{\text{base}}} = \sqrt{T_{\text{top}}/T_{\text{base}}} = 58.5 \quad (23)$$

$$v_{\text{base}} = 5 \frac{m}{s} \cdot 58.5 = 292 \frac{m}{s} \quad (24)$$

$$u_{\text{base}} = \sqrt{\frac{T_{\text{base}}}{\rho}} = 1540 \frac{m}{s} \quad (25)$$

$$F_{\text{transverse, base}} = T_{\text{base}} \sin \theta \quad (26)$$

$$\approx T_{\text{base}} \theta \quad (27)$$

$$\approx T_{\text{base}} \cdot v/u = 1.2 \cdot 10^9 N \cdot 292/1540 = 2.3 \cdot 10^8 N \quad (28)$$

<sup>1</sup> One might ask why energy flux is approximately conserved, but momentum flux is not. Consider an interface between a medium with impedance 1 and a medium with impedance  $1 - \epsilon$ , for small  $\epsilon$ . The reflection coefficient  $R \approx \epsilon/2$ .  $E_{\text{reflected}}/E_{\text{incoming}} \approx \epsilon^2/4$  whereas  $p_{\text{reflected}}/p_{\text{incoming}} \approx \epsilon/2$ . So at a slightly mismatched interface, which produces small partial reflections, conservation of energy flux is correct to first order, and conservation of momentum flux is correct to zeroth order