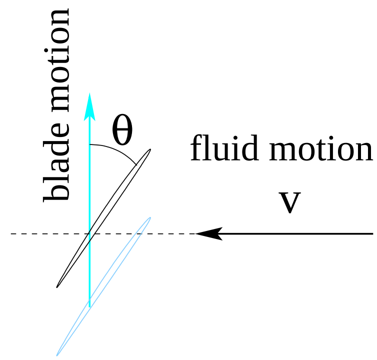

FIG. 1: The blade of the turbine spinning in the flow of an oncoming fluid. As the blade moves, it seems to recede from the oncoming fluid, giving rise to the relative velocity $v_{rel} = v - \omega r tan\theta$.

Tidal energy

I. EFFICIENCY OF A TURBINE

The first step is to estimate how much energy can one extract from water flowing across a turbine.

It is easy to guess what the final expression would be. We need a result that is in units of power, which depends on the flow velocity, v, and on the effective surface area of the turbine, A. We will also need the mass density of the working fluid, ρ. Now, the energy density in a moving fluid is:

$$\frac{E_k}{V} = \frac{1}{2}\rho v^2, \tag{1}$$

therefore, the flux of kinetic energy (kinetic energy flowing through a membrane per unit time per unit area) is:

$$j_E = v \cdot \frac{E_k}{V} = \frac{1}{2}\rho v^3. \tag{2}$$

Finally, a power is obtained by multiplying this by an area:

$$P_{Turbine} \sim A j_E \sim \rho A v^3 \tag{3}$$

Let's refine this result by thinking about a turbine blade as tilting an angle θ from the plane of the turbine (i.e., the plane normal to the fluid flow, see Fig. 1)). Roughly, the fluid quantity dm hitting the turbine loses its momentum normal to the blade. So:

$$dp = dm v_{rel} \cos\theta \tag{4}$$

where v_{rel} is the relative velocity between the incoming fluid and the turbine blade. This produces a force on the blade in the direction of the blades motion which is:

$$dF = (\rho A v) v_{rel} \cos\theta \sin\theta \tag{5}$$

Where $\rho A v$ is dm/dt - the matter current impinging.

And now, we need to add all the torques coming from all impinging liquid:

$$\begin{aligned} \tau &= \int_0^R \Theta R dR \rho v \cdot R \cdot \tfrac{1}{2} v_{rel} \sin(2\theta) \\ &= \tfrac{1}{6}(\Theta R^2) R v v_{rel} \sin(2\theta) \end{aligned} \tag{6}$$

Where Θ is the effective angle that the turbine covers (if it covers a full disk, then $\Theta = 2\pi$). Clearly $\Theta R^2 = 2A$. Also, the power produced by the turbine is:

$$P = \tau\omega = \frac{1}{3}(\omega R) A \rho v \cdot v_{rel} \sin(2\theta) \tag{7}$$

now, it is time to figure out what v_{rel} and what ωR are. If the blade is rotating at angular velocity ω, then the perceived speed of the blade in the direction of the fluid flow is:

$$\omega R \tan\theta$$

Therefore,

$$v_{rel} = v - \omega R \tan\theta \tag{8}$$

This makes the power produced into a function of θ and ω:

$$P(\theta, \omega) = \frac{1}{3} A\rho v(\omega R)(v - \omega R \tan\theta)\sin(2\theta) \tag{9}$$

Let's write this in terms of $u = \omega R \tan\theta$:

$$P(\theta, u) = \frac{1}{3} A\rho v(v-u) 2\cos^2(\theta) u \tag{10}$$

Now we can maximize this expression with respect to θ and ω. Albeit obtains at a non realistic angle of 0 or π, the maximum of $\cos^2\theta$ is 1. Then maximizing over u clearly gives $u = v/2$, and we get:

$$P_{max} \approx \frac{1}{6} A\rho v^3. \tag{11}$$

II. ENERGY FROM THE TIDES IN THE AUSTRALIAN MODEL

In Australia, two stations are producing energy from tidal streams in narrow channels. The info from the web indicates that a small turbine produces $P = 3.5MW$ from a flow velocity of $v = 11m/s$, and a larger turbine of radius $5m$ produces $800kW$ from a flow of $4m/s$. Let's see if this fits.

Starting with the larger turbine:

$$P = \frac{1}{6}\pi(5m)^2 1000kg/m^3 (4m/s)^3 \approx 840kW. \tag{12}$$

Not bad!

From the information given we can guess the radius of the smaller turbine:

$$A = \frac{6P}{1000kg/m^3 \cdot (11m/s)^3} = \frac{6 \cdot 3.5 \cdot 10^6 W}{1.3 \cdot 10^6 kg/s^3} = 15.8m^2 \tag{13}$$

which gives through $A = \pi r^2$ a radius of:

$$r \approx 2m \tag{14}$$

indeed smaller.

III. TIDAL FORCES AT PLAY

A. Estimating the Earth-moon tidal force

Let's calculate the tidal effect from the moon (or the sun) on the earth. First, considering both the Earth and the Moon as point bodies. r_{E-M} is the distance from the Earth to the moon, and R_{cm} is the distance of the Earth from the joint center of mass. Let's also move to the frame rotating with the Earth-moon system (see Fig. 2). In this system, the Earth is in equilibrium:

$$-\frac{Gm_M}{r_{E-M}^2} + \omega^2 R_{cm} = 0 \tag{15}$$

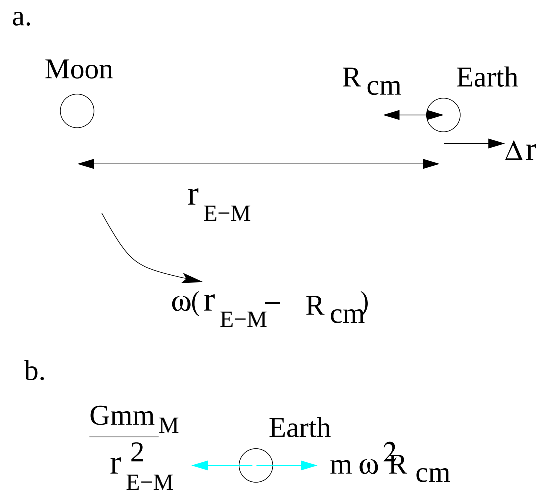

FIG. 2: (a) The Earth-Moon system illustration. Note that both bodies are represented as point objects (otherwise the Earth would contain the center of mass). Δr is defined as the projection of the radius vector of a test mass along the Earth-moon axis, measured from the Earth's center. (b) Free body diagram of the Earth in the rotating frame.

with ω the angular velocity of the frame ($2\pi/28days^{-1}$), and the signs chosen such as positive is away from the center of mass. If we consider a test mass on the line between the Earth moon, which is Δr away from the Earth's center, it would feel an acceleration:

$$a_t = -\frac{Gm_M}{(r_{E-M}+\Delta r)^2} + \omega^2(R_{cm}+\Delta r) = \left(2\frac{Gm_M}{r_{E-M}^3} + \omega^2\right)\Delta r \tag{16}$$

which, we could further simplify by using the equilibrium condition above, to get:

$$a_t = \left(2\frac{R_{cm}}{r_{E-M}} + 1\right)\omega^2 \Delta r. \tag{17}$$

For the Earth-moon system, clearly $R_{cm} \ll r_{E-M}$, so we have:

$$a_t \approx \omega^2 \Delta r. \tag{18}$$

[For the sun-Earth system, we would have to take into account, $R_{cm} \approx r_{E-M}$, which would result in a factor of 3 enhancement of a_t in Eq. (18).]

Eq. (18) can also be made into a tidal potential energy density, by integrating over Δr, and multiplying by the density to obtain:

$$\frac{E_{pt}}{V} = \frac{1}{2}\rho\omega^2\Delta r^2 \tag{19}$$

B. Estimating water velocities due to tidal changes

Let's estimate water velocity ranges and tidal height amplitudes **neglecting the water's inertia and finite response and relaxation time**. If we want to get an estimate of velocities and heights due to potential energy differences, we can always resort to the good old Bernoulli equation:

$$\frac{1}{2}\rho v^2 + \frac{E_p}{V} + p = const \tag{20}$$

In our case it would be:

$$\frac{1}{2}\rho v^2 + \frac{1}{2}\rho\omega^2\Delta r^2 + \rho g h = const \tag{21}$$

with pressure dropping assuming that we are talking about the ocean surface where the pressure is always atmospheric.

To get an estimate for velocity ranges, let's consider $\Delta r \sim 1000km = 10^6 m$. Turning all the tidal energy into kinetic energy clearly gives:

$$v_t^2 = \omega^2 \Delta r^2 = \left(\frac{2\pi}{24 \cdot 28 \cdot 3600s} \right)^2 10^{12} m^2 = (2.6m/s)^2 \tag{22}$$

Indeed a reasonable range.

Now let's estimate the tidal height difference scale. For that we just convert the tidal energy with potential energy.

$$\rho g \Delta h = \frac{1}{2} \rho \omega^2 \Delta r^2 \tag{23}$$

and:

$$\Delta h = \frac{v_t^2}{2 \cdot 9.8m/s^2} = 0.34m \tag{24}$$

If we are brave, we can put a full $\Delta r = 6400km$, which will propel v_t to $15m/s$ and Δh to $11m$. These become unrealistic. They neglect inertia and friction.