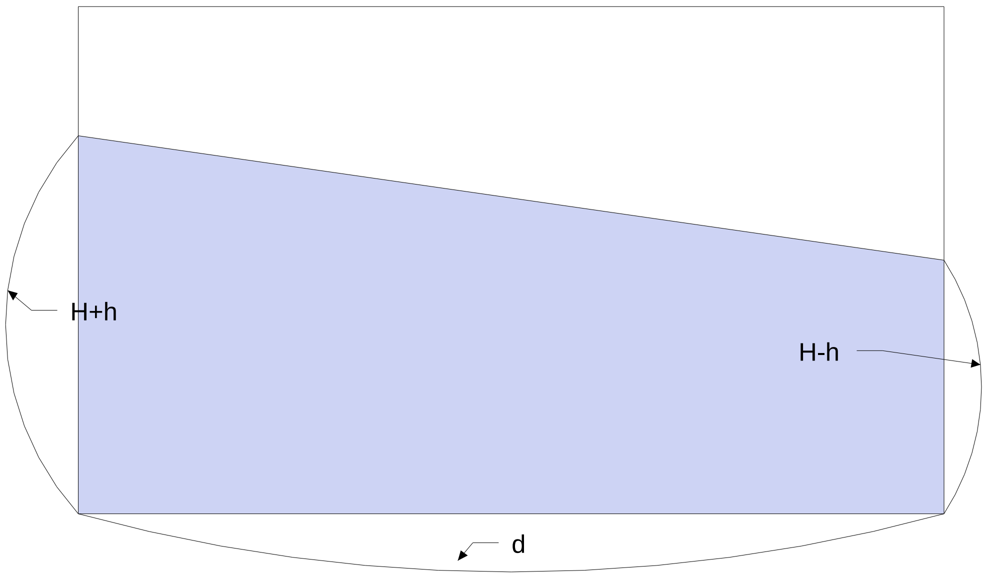
CPL Challenge Solution: How fast is the tide? (written by Gil Refael, Isaac Kim)



I. SIMPLIFIED PROBLEM : RESTORING FORCE ONLY

First step is to study a simplified model of tidal wave. Consider a rectangular box filled water. Width of the box is d and the depth of the water is H. Let us write down the equation of motion when there is a small perturbation. Solving this exactly is an extremely difficult problem, so we shall make some reasonable assumptions that will make the problem easier. We shall assume that there is no friction whatsoever, and we shall assume that the surface of the water is always flat. See FIG.**??**. The height of the water at one side is $H+h$ while at the other side it is $H-h$. What is going to happen is the following. The perturbation h will change in time. When h deviates from 0, the potential energy of the system shall increase. As h changes from nonzero value to 0, the potential energy will be transformed into a kinetic energy. These energies can be expressed in terms of the center of mass of the system.

$$x_{cm} = \int x dm/m \tag{1}$$

$$= \int_{-d/2}^{d/2} x(H + \frac{2h}{d}x)dx/dH \tag{2}$$

$$= \frac{1}{6}h\frac{d}{H}. \tag{3}$$

Similarly, for y-direction,

$$y_{cm} = \int ydm/m \tag{4}$$
$$= \int_{-d/2}^{d/2} y^2 dx/dH \tag{5}$$
$$= \int_{-d/2}^{d/2} (H + \frac{2h}{d}x)^2 dx/dH \tag{6}$$
$$= H + \int_{-d/2}^{d/2} \frac{4h^2}{d^2}x^2 + \frac{4h}{d}xHdx/dH \tag{7}$$
$$= H + \frac{h^2}{3H} \tag{8}$$

Since we assume there is no friction, energy of the system must be conserved.

$$E = \frac{1}{2}mv^2 + mgh \tag{9}$$
$$\approx \frac{1}{2}m\frac{d^2}{36H^2}\dot{h}^2 + mg\frac{h^2}{3H}. \tag{10}$$

From this we can get the equation of motion.

$$\frac{d^2}{72H^2}\ddot{h} + g\frac{h}{3H} = 0 \tag{11}$$

This equation is nothing more than a harmonic oscillator! We can calculate the angular frequency $\omega = \sqrt{24gH}/d$. Therefore the speed is $v = \dot{x}_{cm} = \frac{1}{6}\dot{h}\frac{d}{H} = \sqrt{\frac{2}{3}}\frac{h}{H}\sqrt{gH}$, where h is an amplitude of the oscillation. Suppose that the depth of the water is roughly $10^3 m$ and the amplitude of the oscillation to be $10m$(We shall see in the next section why such numbers are reasonable). The resulting speed becomes $v \approx 1m/s$.

II. RESTORING FORCE + TIDAL FORCE

In the previous section, we realized that the equation of motion is identical to a simple harmonic oscillator. Before we get into the details, let us first think how tidal force will change the solution. First thing that we should notice is that tidal force is periodic: moon rotates around earth. Once we write down the equation of motion, it will look something like the following.

$$m\ddot{x} + kx = a\cos\omega t, \tag{12}$$

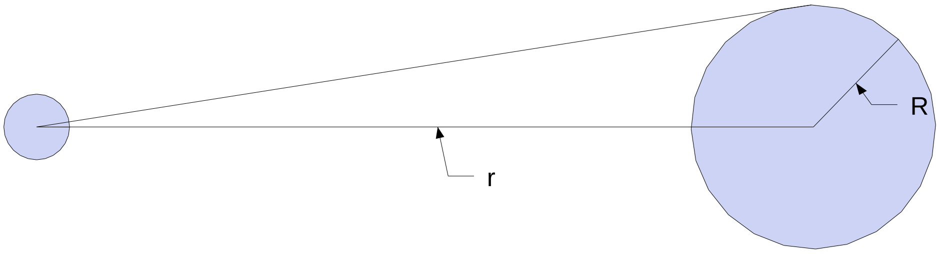
where m is the effective mass, k is the effective 'spring constant' from the restoring force, and RHS being the periodic force originating from the tidal force. Now our task is reduced to finding m, k, a and ω. First look at FIG.**??**.

We shall simplify the problem by assuming that earth is not a sphere, but a disk! We would also like to assume that earth is divided into 4 identical pieces so that the movement of the water in each pieces is identical to each other. Remember that previously, we used a conservation of energy to calculate the tidal speed, but now that we are considering a tidal force, energy is *not* conserved. Instead, we have an external source which is constantly changing the energy of the system. Without the external source, we already showed in the previous section that the equation of motion is

$$m(\frac{d^2}{72H^2}\ddot{h} + g\frac{h}{3H}) = 0, \tag{13}$$

where m is the mass of the water. As one can see in FIG.**??**, we can add the neglected potential energy from the moon.

$$U_m \approx \frac{GM}{r^3}R\int xdm\cos\omega t \tag{14}$$



Here we made couple of assumptions. First, we assumed that the earth-moon distance r is much larger than the radius of the earth. In this limit, the tangential component of the gravity is roughly $\frac{GM}{r^3}R\cos\omega t$. The potential energy of the quarter segment, up to a constant term, can be obtained by summing up each infinitesimal pieces of water. The driving force on the system is now $\frac{\partial}{\partial h}U_m$. Therefore the equation of motion is

$$dH(\frac{d^2}{72H^2}\ddot{h} + g\frac{h}{3H}) = \frac{GM}{r^3}R\cos(\omega t)\frac{d^2}{3} \tag{15}$$

Solution of this equation is

$$h = \frac{GMR}{r^3}\frac{2\sqrt{2}}{\pi}(g/3H - d^2/72H^2\omega^2)^{-1}\frac{d}{3H}\cos\omega t \tag{16}$$

Putting in the numbers, we get $v = \frac{d}{6H}\dot{h} = \frac{d}{6H}h\omega$

$$h \approx (6400/405000)^3 \times 10 \times (10/3000 - (6400 \times 2\pi/24/3600)^2/72) \times \frac{6400}{3} \approx 240m. \tag{17}$$

From this, we can estimate $v \approx 17m/s$. There are couple of things that I would like to comment. First, if you take the $\omega \to 0$ limit, you will find $h \approx 10m$, which justifies the number that we used in the previous section. Second, if you do the actual calculation, denominator part, $g/3H - d^2/72H^2\omega^2$ miraculously cancel out each other to be a small number. It is this coincidence that makes the speed estimate in this section much larger. More likely, when you plug in other numbers for the depth of the water, you will end up with a smaller tidal speed.