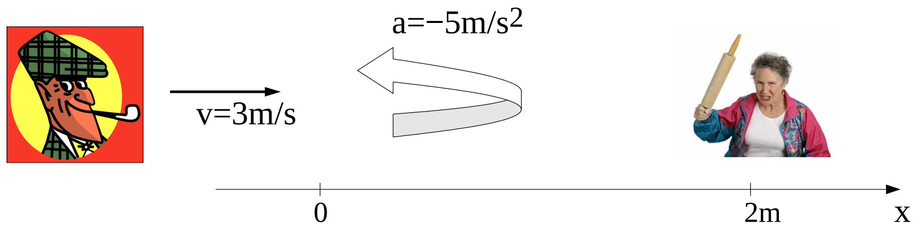

FIG. 1: We set the origin $x = 0$ at the house entrance, and the time origin $t = 0$ as the time in which Paul enters the house. **Put as much of the info given on the diagram!**

Section 2 example problems and discussion

1. **Encounters with the law.**

Paul comes home from work, and through the open house door, at a distance of $2m$ from it inside, he sees whom he thinks is his wife. Happy, he runs into her arms with velocity of $3m/s$. But as he enters the house he realizes it was actually his mother in law (MIL), and immediately tries to run away, back to work. He decelerates (accelerates backwards) by $5m/s^2$, and does not stop this acceleration until he exits the house.

(a) Plot Paul's motion.

(b) How much time does it take Paul to leave the house?

(c) How close does Paul get to his menacing MIL?

Solution:

Before starting to solve the problem, we must make a good diagram of what's going on (see Fig 1)

Now that we have the diagram, we need to think of the equations of motion. In any segment of time, in which the object we follow moves in a uniformly accelerated motion, we can write:

$$x(t) = x_{t_0} + v_{t_0}(t - t_0) + \frac{1}{2}a(t - t_0)^2 \quad (1)$$

Note that t_0 is the time at which x is measured to be $x(t_0) = x_{t_0}$ and the velocity is $v(t_0) = v_{t_0}$, and it doesn't have to be in the beginning of the segment, as you shall see right below.

Let's start constructing the equations that control Paul's motion. The uniformly accelerated segments are:

$$\begin{array}{c} (1)\, t < 0: \quad v = 3m/s \quad a = 0 \\ (2)\, 0 < t < t_{exit}: \quad v = 3m/s + at \quad a = -5m/s^2 \\ (3)\, t > t_{exit}: v = -3m/s \quad a = 0 \end{array} \quad (2)$$

The motion in segment (1) has a constant velocity. We can utilize Eq. (1) for this case by choosing: $t_0 = 0$, $x_{t_0} = x_0 = 0$. Then we write for $t < 0$:

$$x(t) = 3m/s \cdot t \quad (3)$$

Note that since $t < 0$ indeed $x < 0$ in this segment, which translates to Paul being out of the house.

Segment (2) is similarly simple:

$$x(t) = 3m/s \cdot t + \frac{1}{2}(-5m/s^2)t^2 \quad (4)$$

This is a quadratic equation, and it is always good to work with lower order equations. So let's construct also the velocity in this segment. It can be obtained either through common sense by $v(t) = v_{t_0} + a(t - t_0)$ or by differentiating the $x(t)$ curve:

$$v(t) = \frac{dx}{dt} = 3m/s + (-5m/s^2)t \quad (5)$$

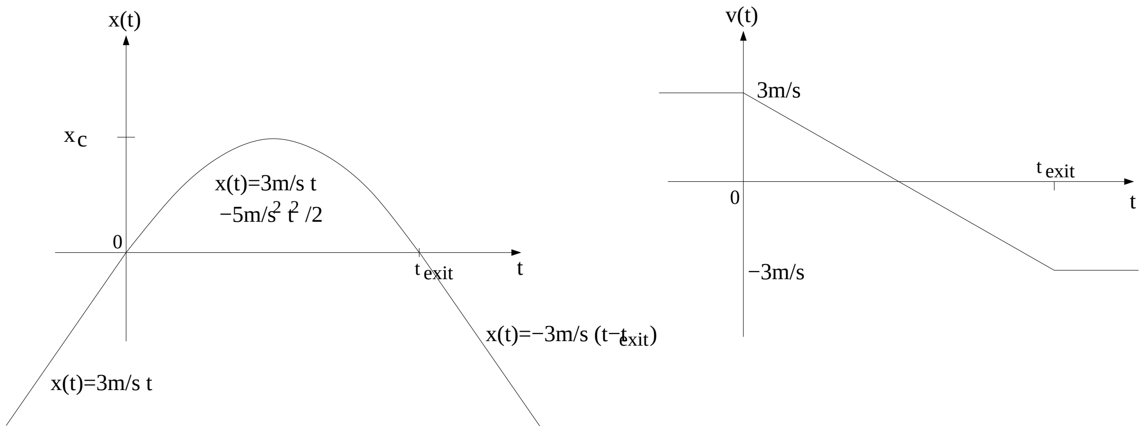

FIG. 2: $x(t)$ and $v(t)$ of the motion described in the text. When plotting the curves, put as much information as you can regarding features of the graph.

Segment (3) is like segment (1) except moving backwards. First, how do we know that $v_{t_{exit}} = -3m/s$? We know this because Paul's motion away from his mother in law is a mirror image of his motion towards her. The equation of motion is then:

$$x(t) = 0 - 3m/s \cdot (t - t_{exit}) \tag{6}$$

Now we can answer all questions in leisure.

(a) Plot: See Fig. 2.

(b) When writing a solution, clarify what it is you are seeking: Here we have $t_{exit} =?$. We can do this in two ways. First, we can just ask, when is Paul at the door?

$$x(t_{exit}) = 0 = 3m/s t_{exit} - \frac{1}{2} 5m/s^2 t_{exit}^2 = t_{exit}\left(3m/s - \frac{1}{2} 5m/s^2 t_{exit}\right) \tag{7}$$

This has two solutions: $t_{exit} = 0$, and $t_{exit} = \frac{2v_0}{|a|} = \frac{6m/s}{5m/s^2} = 1.2s$. Clearly, the first solution is just the time Paul enters the house, and the second is the one we are looking for.
The second way of getting this is by saying that when Paul leaves the house, his velocity is equal in value and opposite in sign to his initial velocity. We can then use the function $v(t)$ in segment (2) and find:

$$v(t_{exit}) = v_0 + a t_{exit} = -v_0 \tag{8}$$

which gives:

$$t_{exit} = \frac{2v_0}{|a|} = \frac{6m/s}{5m/s^2} = 1.2s. \tag{9}$$

Exactly the same as before, but without having to think of the quadratic equation.

(c) Her we have $2m - x_c =?$ where x_c indicates x at closest approach to the MIL. x_c is obtained at the top of the parabola that $x(t)$ gives in segment (2). Technically speaking, we can find this by asking what is the maximum of $x(t)$, which we do by asking where is $dx/dt = 0$ (at a maximum or minimum a function is going to have slope zero). But this just translates to a the velocity vanishing. The time at which it happens we find from $v(t)$:

$$\begin{aligned} v(t_c) &= v_0 + a t_c = 0 \\ t_c &= -\tfrac{v_0}{a} = \tfrac{1}{2} t_{exit} = 0.6s \end{aligned} \tag{10}$$

Easy. How could we have known that $t_c = \frac{1}{2} t_{exit}$? From the fact that going towards the MIL is a mirror image of running away. What about x_c? We can plug t_c in $x(t)$ for segment (2):

$$x(t_c) = v_0 t_c + \frac{1}{2} a t_c^2 \tag{11}$$

but this we can massage into something simpler using what we already know about $t_c = -v_0/a$. We write:

$$x(t_c) = t_c(v_0 + \frac{1}{2} a t_c) = t_c(v_0 - \frac{1}{2} v_0) = \frac{1}{2} v_0 t_c = 1.5m/s \cdot 0.6s = 0.8m \tag{12}$$

And closest distance:

$$d_{minimum} = 2m - 0.8m = 1.2m \tag{13}$$

Too close for comfort.

2. **Fusion propulsion ship.**

In the year 2523 the starship Enterprise was equipped with a hydrogen collection and fusion propulsion engine. This means that as it roams through space, it collects interstellar hydrogen, and uses it to propell the ship through a fusion reaction. The maximum accelration of the Enterprise is therefore proportional to the rate of hydrogen collection, which is proportional to its velocity (neglecting relativistic effects):

$$a_{max} = \gamma v$$

with $\gamma = 0.1 s^{-1}$. The regular engines of the Enterprise get it to velocity $v_0 = 10^4 m/s$, at which point (set time $t = 0$) it turns its fusion propulsion on at maximum output.

(a) What is the velocity of the Enterpise as a function of time? (Neglect relativistic effects).

(b) When does it reach warp speed 1/2 (ie half the speed of light, where relativistic effects set in)?

(c) When does it reach twice its initial velocity?

answer:

(a) Once the fusion engine kicks in, and the starship accelerates at its maximum rate. This implies a rather curious relationship between its velocity and its acceleration, which is the time-derivative of its velocity:

$$\frac{dv}{dt} = \gamma v. \tag{14}$$

Such a relationship is called a **differential equation**. Differential equations are usually hard to solve, and each equation may require a different solution technique. You have already solved simpler differential equations. For example, a uniformly accelerated motion is:

$$\frac{dv}{dt} = a \tag{15}$$

by asking what function gives a constant derivative, we come up with the answer: constant times t. This immediately give the answer: $v = at + v_0$, where we added v_0 to match boundary conditions, noticing that the derivative of a constant is zero. Then we went further and solved the differential equation (assuming $t_0 = 0$):

$$\frac{dx}{dt} = v_0 + at \tag{16}$$

which we can do by asking: What function has derivative t? the answer is $t^2/2$. We can also directly solve it by multiplying both sides by dt and integrating:

$$\int_{x_0}^{x(t)} dx = \int_{t_0=0}^{t} (v_0 + at) dt = v_0 + \frac{1}{2} a t^2. \tag{17}$$

Let's get back to our equation, Eq. (14). The crucial element is that the derivative of $v(t)$ is proportional to $v(t)$ itself. We can ask: What function equal to its derivative? The answer is the function e^t:

$$\frac{d}{dt} e^t = e^t \tag{18}$$

What function would give γ times its derivative? Note that in the above, we can just redefine time, like changing the units with which we measure time:

$$t \to \gamma t \tag{19}$$

which also implies

$$dt \to \gamma dt \tag{20}$$

and we would have:

$$\frac{d}{\gamma\ dt} e^{\gamma t} = e^{\gamma t} \tag{21}$$

which is exactly the equation Eq. (14)!
So if we set:

$$v(t) = e^{\gamma t} \tag{22}$$

we fulfilled Eq. (14). Is this the only way to fulfill it? No. Multiplying $v(t)$ by any constant would also yield an honest solution, $v(t) = const \cdot e^{\gamma t}$ of Eq. (14). The differential equation must be accompanied by an **initial condition**, which in this case is:

$$v(0) = v_0. \tag{23}$$

plugging this into the guess solution above, and we find:

$$v = v_0 e^{\gamma t} \tag{24}$$

(b) Now everything is easy. To find when the velocity is half of the speed of light, we solve:

$$c/2 = v_0 e^{\gamma t_{c/2}}. \tag{25}$$

To get at the unknown time $t_{c/2}$ we need to extract the argument of the exponent function. We do this using the *inverse* function of the exponent: the logarithm function:

$$\ln e^x = x \tag{26}$$

Recall the properties of the ln function:

$$\ln(a \cdot b) = \ln a + \ln b, \qquad \ln \frac{a}{b} = \ln a - \ln b, \qquad \ln a^n = n \ln a.$$

Applying the ln to both sides of (25) we get:

$$\begin{aligned} \ln \tfrac{c}{2} &= \ln v_0 + \gamma t_{c/2} \\ \frac{\ln \frac{c}{2v_0}}{\gamma} &= t_{c_2} = 96s \end{aligned} \tag{27}$$

(c) Speed doubling. To find the speed doubling time all we need to do is is write $v(t) = 2v_0$: $2v_0 = v_0 e^{\gamma t_D}$. Cancelling v_0 and taking the log on both sides, we obtain:

$$t_D = \frac{\ln 2}{\gamma} \tag{28}$$

A quick note is in order regarding the acceleration at t_D. Since the accelration is $a = dv/dt = \gamma v$ we also have:

$$\frac{a(t_D)}{a(0)} = \frac{v(t_D)}{v_0}, \tag{29}$$

i.e, the rate of change of the velocity is also half its initial rate at t_D. This should be useful when you think of ch 3 problem 13.