Ph 127b - Problem set 2

1. Water and vapor as a Van der Waals gas
The critical pressure and temperature of water are $p_c = 218\text{Atm}$, $T_c = 647\text{K}$.

(a) Express the Van der Waals parameters $a$ and $b$ in terms of $p_c$ and $T_c$.
(b) By hook or by crook (i.e., using numerics, or justifiable approximations), find the coexistence pressure $p_c(T)$ at $T = 373\text{K}$, as predicted by the Van der Waals theory.
(c) Show that the Latent heat of vaporization for a Van der Waals gas is given by:
\[
L_{\text{lg}} = p(v_g - v_\ell) - a \left( \frac{1}{v_g} - \frac{1}{v_\ell} \right).
\]

What does this predict for water-steam transition at atmospheric pressure? How does this compare to the true value? Feel free to look up any constants necessary for solving this problem.

2. Ising lattice on a bipartite lattice.
A bipartite lattice is a lattice whose sites can be divided into two groups such that lattice sites in group A are only nearest neighbors of sites from group B and vice versa (for example, the square lattice is bipartite, and shown in Fig. 1).

In class, we showed that the inversion transformation $R\sigma = -\sigma$ transforms the Ising Hamiltonian:
\[
\hat{H} = -\sum_{\langle i, j \rangle} J\sigma_i \sigma_j - h \sum_i \sigma_i
\]

(2)
to the same Hamiltonian with
\[
h \to h' = -h.
\]

(3)
Show that for the Ising model on a bipartite lattice, a similar transformation transforms the Ising model (2) to a model with a Hamiltonian as in (2), but with:
\[
J \to J' = -J
\]

(4)
and:
\[
h \to h'_i = \begin{cases} 
  h & i \in A \\
  -h & i \in B 
\end{cases}
\]

(5)
where the magnetic field in the new Hamiltonian depends on which lattice site it operates.
The solution of this problem constitutes a proof that on a bipartite lattice a ferromagnetic Ising model is equivalent to the anti-ferromagnetic Ising model.
FIG. 2: Decorated Ising model. Spins live on the drawn vertexes and interact via the black lines drawn with their nearest neighbors.

3. Susceptibility and correlations in the Ising model.

(a) Using the definition of the magnetic susceptibility (per length), $\chi = -\frac{\partial^2 F}{\partial h^2}$, show that:

$$\chi = \frac{1}{L \cdot T} \sum_{i,j} \left( \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle \right)$$  \hspace{1cm} (6)

(b) Use this formula, and the correlation function derived in class, to obtain the susceptibility of the 1-d Ising model with free boundary conditions in the thermodynamic limit (i.e. when $L \to \infty$). Assume the interaction between nearest neighbors is $J$, and the temperature is $T$. (For this part you will need the correlation calculation that will be discussed on Tuesday Jan 21st)

4. Decorated Ising model. Consider an Ising model with Hamiltonian:

$$\hat{H} = -J \sum_{(i,j)} \sigma_i \sigma_j$$  \hspace{1cm} (7)

where the spins live on the black dots in the chain in Fig. 2. The spins interact with nearest neighbors only along the drawn black lines.

(a) What is the partition function of this system?

(b) Find the heat capacity of this system.

(For the next two parts you will need a trick that will likely be taught on Friday Jan 23rd)

(c) What is the correlation function between two spins $i$ and $j$? Separate your answer to three cases - (a) both spins on the backbone, (b) both spins on the sides, (c) one spin on the backbone and one isn’t. Assume both spins are away from the sides of the chain.

(d) Find the magnetic susceptibility per length of this system in the thermodynamic limit (Hint: use the results for the correlations).