Problem Set #5

(1) a) \[ C = \frac{-2^{a} f}{2h^2} - b^{a} h^{-d} f''(tb^d + h b^d y_h) \]

A key observation is that we are at a critical point, so everything is independent of the scale \( b \). So we can choose \( b \) to be whatever we want.

Let's choose \( b = h^{-\frac{1}{d}} \). Then the 2\(^{nd} \) argument to \( f'' \) vanishes is one, and we have

\[ C = -(h^{-\frac{1}{d}})^{2^{d-1}} f'' \left( \frac{t}{h^{\frac{1}{d}}} \right) \]

Clearly, \( \beta = \frac{1}{vy_h} \).

\[ C = -\left( \frac{1}{h^{2-d}} \right) \beta f'' \left( \frac{t}{h^\beta} \right) = -\left( \frac{2-vd}{h^\beta} \right) \left( \frac{2-vd}{h^\beta} \right) f'' \left( \frac{t}{h^\beta} \right) \]

Let \( g(t) = x^{2-vd} f''(x) \), so \( \alpha = 2-vd \), and we get

\[ C \alpha = g \left( \frac{t}{h^\beta} \right), \text{ as required} \]

b) \[ C = t^{\alpha+1} g \left( \frac{t}{h^\beta} \right) \]

\[ \frac{2C}{2h} = t^{\alpha+1} \left( \frac{-\beta}{h^\beta} \right) g' \left( \frac{t}{h^\beta} \right) = t^{\alpha+1} \left( \frac{-\beta}{h^\beta} \right) \left( \frac{t}{h^\beta} \right)^{\alpha+1} g' \left( \frac{t}{h^\beta} \right) \]

\[ = g_a \left( \frac{t}{h^\beta} \right) \]

∴ a suitable scaling form is \( C \alpha + \beta = g_a \left( \frac{t}{h^\beta} \right) \)

c) \[ \frac{2C}{2h} = \frac{-2^{a} f}{2^{d+2} h} = \frac{-2^{d}}{2^{d+2}} m \]
(2a) As can be clearly seen in the figure, the new lattice constant is $\sqrt{5}$ longer than the old lattice constant, so $b = \sqrt{5}$

b) The answer to this is different depending on if $S_i$ is on the edge or center of the block. Then

we have

$$\langle S_i \rangle_M = \sum_{S_{i+1}} e^{-\beta H} S(M - sgn(S_i+S_{i+1})) S_i$$

$$= \sum_{S_{i+1}} e^{-\beta H} S(M - sgn(S_i+S_{i+1})) S_i$$

Where, as in the notes, we have only considered one block.

Let's consider $M=+1$ first. There are 5 terms which contribute to the sum

$$+ + + + +$$

$$e^{4\beta S} e^{2\beta S} e^{2\beta S} e^{-\beta S} e^{-\beta S}$$

weight

$$1 \quad 4 \quad 6 \quad 1 \quad 4$$

factor in $Z$

$$1 \quad 4 \quad 6 \quad -1 \quad -4$$

factor if $S_i$ in center

$$1 \quad 2 \quad 0 \quad 1 \quad 2$$

factor if $S_i$ on edge

Also note that the $M=-1$ case is the same but for an overall $-sgn$.

$S_i$ in center: $\langle S_i \rangle_M = \frac{2\sinh (4\beta S) + 8\sinh (2\beta S) + 6}{2\cosh (4\beta S) + 8\cosh (2\beta S) + 6}$

$S_i$ on edge: $\langle S_i \rangle_M = \frac{2\cosh (4\beta S) + 4\cosh (2\beta S)}{2\cosh (4\beta S) + 8\cosh (2\beta S) + 6}$
c) Each block is connected by 3 bonds

Using the 1st order cumulant expansion (as in class)
we get \( V' = -3J \tilde{M}_i \tilde{M}_j \langle s_i s_j \rangle \) (eq. 328)

\[
(\beta J) = 3 \beta J \left( \frac{2 \cosh(4\beta J) + 4 \cosh(2\beta J)}{2 \cosh(4\beta J) + 8 \cosh(2\beta J) + 6} \right)^2
\]

\( d) \) Obviously there are fixed points at \( \beta J = 0, \beta J = \infty \)

There is also one where \( \langle \beta J \rangle = \rho J \).

Let \( x = \cosh(2\beta J) \)

Note that \( \cosh(4\beta J) = 2x^2 - 1 \)

Some simplification gives

\[
\frac{2x^4 + 2x - 1}{2x^4 + 4x + 2} = \frac{1}{\sqrt{3}}
\]

Giving quadratic eqn \( 2(\sqrt{3} - 1)x^4 + 2(\sqrt{3} - 2)x - (\sqrt{3} + 2) = 0 \)

\[
x = \frac{\sqrt{3} - 1 \pm \sqrt{24 + 10\sqrt{3}}}{4} \approx 1.790 \quad (\text{approx. } 0.593)
\]

\( e) \) Using Eq. 350 from the notes, week 9-16

\[
V = \frac{\ln b}{\ln \left( \frac{2t(\beta J)}{\beta J} \right)} \bigg|_{\beta J = 0.593}
\]

\[
V = 1.11
\]