VIII. EXACT SOLUTION WITH TRANSFER MATRICES

A slightly more useful way of solving the 1-d Ising model is the method of transfer matrices. This method can very often be employed in other 1-d systems. It applies to the periodic Ising model:

$$Z = \sum_{\{S_i\}} e^{\beta J \sum_{i=1}^{L-1} S_i S_{i+1} + \beta J S_1 S_L}$$  \tag{86}

We write this in a quantum mechanical way:

$$Z = \sum_{\{S_i = \pm 1\}} \langle S_1 | e^{\beta J S_1 S_2 + \beta h (S_1 + S_2)/2} | S_2 \rangle \langle S_2 | e^{\beta J S_2 S_3 + \beta h (S_2 + S_3)/2} | S_3 \rangle \ldots \langle S_L | e^{\beta J S_L S_1 + \beta h (S_L + S_1)/2} | S_1 \rangle$$  \tag{87}

But note that the combination

$$\sum_{S_1 = \pm 1} |S_2\rangle \langle S_2| = 1$$  \tag{88}

is the identity matrix in the Hilbert space of a single spin. And, next, the Combination:

$$\langle S_1 | e^{\beta J S_1 S_2 + \beta h (S_1 + S_2)/2} | S_2 \rangle$$  \tag{89}

is now a 2X2 matrix where the 11 entry corresponds to up-up, the 22 down-down, and so forth. This matrix is:

$$T = \langle S_1 | e^{\beta J S_1 S_2 + \beta h (S_1 + S_2)/2} | S_2 \rangle = \begin{pmatrix} e^{\beta h + \beta J} & e^{-\beta J} \\ e^{-\beta J} & e^{-\beta h + \beta J} \end{pmatrix}$$  \tag{90}

This is called the transfer matrix of the Ising model.

Now the partition function is just a trace over a product of these guys. Hence:

$$Z = tr \left[ \begin{pmatrix} e^{\beta h + \beta J} & e^{-\beta J} \\ e^{-\beta J} & e^{-\beta h + \beta J} \end{pmatrix}^L \right]$$  \tag{91}

This looks tough, except that here we get a bonus - a trace of product of matrices in independent of basis. Hence, if we diagonalize $T$:

$$STS^{-1} = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}$$  \tag{92}

then:

$$Z = \lambda_+^L + \lambda_-^L$$  \tag{93}

Let’s find these eigenvalues:

$$(\lambda - e^{\beta h + \beta J})(\lambda - e^{-\beta h + \beta J}) = e^{-2\beta J}$$  \tag{94}

This yields:

$$\lambda_{\pm} = e^{\beta J} \left( \cosh \beta h \pm \sqrt{\sinh^2 \beta h + e^{-4\beta J}} \right)$$  \tag{95}

The resulting free energy for a large system is then:

$$F = -T \ln Z = -TL \ln (\max \{\lambda_+, \lambda_-\})$$  \tag{96}

The transfer matrix gives us a new way of asking about phase transitions - In order to get a singular behavior of the partition function, the eigenvalues of the transfer matrices should act funny. But we see that they are both analytic as a function of both $T$ and $h$. The square root is always positive. But there is another way that the partition function can act funny - if the two eigenvalues cross at a certain point. If they do, suddenly:

$$F \approx -TL \ln \lambda_+ \rightarrow F \approx -TL \ln \lambda_-$$  \tag{97}

which will give a first order phase transitions, since there will be a wedge in the free energy, and therefore a jump in a first order derivative.

But here comes to our help, or to our demise, a theorem - Perron’s thm: Given a finite matrix with positive definite entries, the largest eigenvalue is
1. real and positive,
2. non-degenerate,
3. analytic function of the matrix elements (this means that it is infinitely differentiable).

The non-degeneracy precludes level crossings.

**A. Susceptibility**

Let’s get the susceptibility of the Ising model - since now it is really easy.

In the thermodynamic limit we have:

\[ F = -TL \ln \lambda_+ = -TL \ln e^{\beta J} \left( \cosh \beta h + \sqrt{\sinh^2 \beta h + e^{-4\beta J}} \right) \]  

(98)

Now, the magnetization is:

\[ -\frac{\partial F}{\partial h} = TL \frac{1}{\lambda_+} \frac{\partial \lambda_+}{\partial h} = LT \frac{\beta}{e^{\beta J} \left( \cosh \beta h + \sqrt{\sinh^2 \beta h + e^{-4\beta J}} \right)} \left( \sinh \beta h + \frac{\sinh \beta h \cosh \beta h}{\sqrt{\sinh^2 \beta h + e^{-4\beta J}}} \right) \]  

(99)

This simplifies nicely to:

\[ L \sinh \beta h \]  

(100)

and in the limit of \( h \to 0 \) we have:

\[ \approx Le^{2\beta J} \frac{h}{T} \]  

(101)

So, indeed, we find critical behavior as the temperature goes to zero, the susceptibility has an essential singularity:

\[ \chi = Le^{2\beta J} \frac{1}{T} \]  

(102)

As \( T \gg J \) we obtain Curie for free spins, \( \chi \sim 1/T \). Note that this completely agrees with the connection between the susceptibility and the correlation length.

**B. Correlation length using the transfer matrices**

Now, to calculate the correlation length using the transfer matrices we need to work just a little bit more. Here is how it goes. Let’s go back to the expression for the partition function as a trace formulae:

\[ Z = \sum_{\{S_i=\pm\}} \langle S_i \rangle e^{\beta JS_iS_{i+1} + \beta H_{i, i+1} / 2} \langle S_1 \rangle \ldots \langle L \rangle e^{\beta JS_{L-1}S_L + \beta H_{j, j+1} / 2} \langle S_0 \rangle = tr T^L \]  

(103)

If we want to calculate the correlation between \( S_i \) and \( S_j \), we need to know how much of the probability in this large sum belongs to parallel \( S_i \) and \( S_j \), and how much to antiparallel. But that is easy:

\[ \langle S_i S_j \rangle = \frac{1}{Z} \sum_{\{S_i=\pm\}} \langle S_i \rangle e^{-\beta H_{i, i+1}} \langle S_{i+1} \rangle \ldots \langle S_L \rangle e^{-\beta H_{j, j+1}} \]  

(104)

In terms of the transfer matrices, each of the spin insertions is a matrix, \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \). Therefore this evolves to:

\[ \Rightarrow \frac{1}{Z} tr \left( T^{i-j} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} T^{j-i} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} T^{L-j+1} \right) \]  

(105)
This is an exact expression, but we need to disentangle it. We again recall that $STS^{-1} = \text{diag} \{ \lambda_+, \lambda_- \}$. All of us know how to find $S$ from the formulae for $T$ and solve this conundrum. Let me show you a neat way to do it, though.

When the magnetic field is zero we have:

$$T = \begin{pmatrix} e^{\beta J} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J} \end{pmatrix} \quad \text{(106)}$$

and we write it in terms of Pauli matrices. Why Pauli matrices? Because as physicists, these are the matrices we know best. For instance, it doesn’t take a lot of observation to see that:

$$= 1 e^{\beta J} + \sigma_x e^{-\beta J} \quad \text{(107)}$$

where $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We want to diagonalize this matrix - nothing has ever been easier. For diagonalization purposes we can ignore the diagonal part - intimidated by our prowess in 2x2 matrix diagonalization it is already in the desired form. The seemingly recalcitrant $\sigma_x$ part though, is like a term in the hamiltonian of a spin-1/2 spin of a magnetic field that points in the x direction.

$$\hat{H} = e^{-\beta J} \hat{x} \cdot \sigma \quad \text{(108)}$$

How do we diagonalize this hamitonian? We just rotate the x direction to the z direction. This is obtained with the rotation transformation:

$$S = e^{i \frac{\pi}{2} \sigma_y} \quad \text{(109)}$$

a rotation by ninety degrees about the y axis, lands the x-axis on the z-axis. This implies that the diagonal form is:

$$\Lambda = S^{-1} T S = 1 e^{\beta J} + \sigma_z e^{-\beta J} = \begin{pmatrix} e^{\beta J} + e^{-\beta J} & 0 \\ 0 & e^{\beta J} - e^{-\beta J} \end{pmatrix} \quad \text{(110)}$$

indeed! These are the $\lambda_-^\text{e}^\text{+}$ and $\lambda_-^\text{e}^-$. Now, the correlations we can write as:

$$\rightarrow \frac{1}{Z} \text{tr} \Lambda^{j-1} S^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} S \Lambda^{j-1} S^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} T^{L-j+1} S \quad \text{(111)}$$

But now the inserted spin matrices are just:

$$S^{-1} \sigma_z S.$$

This just rotates the z-axis by ninety degrees about the y-axis, and yields - right! the -x-axis! So we need to calculate:

$$\rightarrow \frac{1}{2} \text{tr} \left( \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} \right)^{i-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left( \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} \right)^{j-i} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left( \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} \right)^{L-j+1}$$

$$= \frac{1}{\lambda_+^L + \lambda_-^L} \text{tr} \left( \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} \right)^{L-[j-i]} \left( \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix} \right)^{j-[i]} \quad \text{(112)}$$

In the last step you can see that the sandwich with $\sigma_x$ just gives you an inversion of the lambda’s and the final answer is:

$$\frac{\lambda_-^{L-[j-i]} \lambda_+^{j-[i]} + \lambda_-^{L-[j-i]} \lambda_+^{j-[i]}}{\lambda_+^L + \lambda_-^L} \quad \text{(113)}$$

and in the thermodynamic limit:

$$\left( \frac{\lambda_-}{\lambda_+} \right)^{[i-j]} = \tanh^{[j-i]} \beta J \quad \text{(114)}$$

fewww.. seemingly lots of work, but actually, not so bad, was it? This yields the same result for $\xi$ as the free boundaries.

In fact, I didn’t need to work too hard to just get the correlation length - the only thing that the insertion could do is mix up to eigen-values of the transfer matrix, therefore quite generally, the largest correlation length in a 1-d problem is:

$$e^{-1/\xi} = \frac{\lambda_1}{\lambda_0} \quad \text{(115)}$$

the second largest divided by the largest eigenvalue of the transfer matrix.
IX. MEAN FIELD THEORY

What we have done so far, is to roughly discuss symmetry breaking and phase transitions, and proceeded to show that in 1-d this all story is a bit of a problem, and that the only time we can have a broken symmetry or a phase transition is when $T = 0$, or at higher dimensions. Along the way we learned a couple of methods in statistical mechanics of interacting bodies.

But generically, I told you and you agreed, we have phase transitions. Indeed, above the lower critical dimension, we do. How do we deal with those, generically? The simplest answer is mean-field theory. We will start with its simplest illustration, and work it up into the a general framework - the Landau theory.

Let’s consider an Ising ferromagnet on an arbitrary lattice, with coordination number $z$. The Hamiltonian is:

$$\hat{H} = -J \sum_{\langle ij \rangle} S_i S_j - h \sum_i S_i$$

This is an interacting theory, and the best we can do is to reduce it into a non-interacting theory. Sounds simple. We begin with a hard problem, and all we need to do is just to make an easy problem out of it... There is always a way to do it - ignore complications. In this case the trick is to assume that each spin, instead of seeing vital opinionated neighbors, it just sees an average buzz of spins that seems like some effective average magnetic field:

$$-J \sum_{\langle ij \rangle} S_i S_j \rightarrow -JS_i \sum_j S_j \approx -JzS_i \langle S \rangle = -JzmS_i$$

So the effective Hamiltonian for each spin becomes:

$$\hat{H}_{eff} = -(Jzm + h)S_i$$

This implies a partition function for each particle:

$$Z = 2 \cosh(\beta Jzm + \beta h)$$

and an average magnetization:

$$m = \tanh(\beta(Jzm + h))$$

Ahhh! here is the point. The magnetization $m$ appears on both sides of the equation, it is a self-consistent equation. Now, how do we solve equations like that? One way is to use mathematica. Although this is the way I tend to prefer, in the interest of a good education I need to show you the good old computerless methods - graphical solution of transcendental equations. Let’s focus on $h = 0$.

The important thing is that the slope of the tanh at 0 is 1 with respect to its argument. Therefore, it seems that when $h = 0$ we have the following, when

$$m = 0 \quad T > Jz$$

But when $T < Jz$ $m$ has a nonzero solution as well. To find it we expand the tanh to third order:

$$m = \beta Jzm - \frac{1}{3} (Jz/T)^3 m^3$$

Solving for $m$:

$$m \approx \pm \sqrt{3}(T/Jz)^{1/2} \left( 1 - \frac{T}{Jz} \frac{T}{T_c} \right) \sqrt{|t|}$$

A. Critical exponents in the MF approximation

As I emphasized before, the most important thing to measure in a phase transition are the universal quantities. First and foremost, the critical exponents - the greek letters.

From the above we immediately have $\beta$:

$$|m| \sim |t|^{0.5}$$
so:

\[ \beta = 1/2 \]  

What about susceptibility? Let’s put in the magnetic field to first order in the self consistent equation:

\[
m(1 - T/T_c) \approx h/T - \frac{1}{3} \left( \frac{T_c}{T} \right)^2 \left( \frac{T_c}{T} m^3 + m^2 h \right) \]

This implies:

\[
m \sim \frac{h}{T(1 - T/T_c)} \]

so:

\[
\chi \sim \frac{1}{|z|} \]

and \( \gamma = 1 \).

Here there was a bit of a subtlety that I swept under the rug. I neglected the higher order terms in Eq. (126). This assumes that \( m = 0 \) when \( h = 0 \). But when \( T < T_c \), this is not the case. In fact, Eq. (127) would then imply a negative susceptibility. When we neglected the higher order terms, we essentially limited ourselves to the \( m = 0 \) solution of the mean-field self-consistent equation for the magnetization. Now we see that below \( T_c \), the \( m = 0 \) solution possesses a negative susceptibility. A no-no. Just like the compressibility going negative in the non-physical solution of the Van der Waals equation of state, this negative susceptibility implies that the \( m = 0 \) solution at \( T < T_c \) is unphysical. The \(|m| > 0 \) solution is. How would we find the susceptibility there? Try it out. Write:

\[
\frac{\partial}{\partial h} \left( m(T/T_c - 1) + h/T - \frac{1}{3} \left( \frac{T_c}{T} \right)^2 \left( \frac{T_c}{T} m^3 + m^2 h \right) \right) = 0
\]

and extract \( \chi = \frac{\partial m}{\partial h} \) from this implicit equation, using the non-zero solution for \( m \) that we obtained before.

Last for now, we can get the equation of state - and \( \delta \). Set \( t = 0 \) and we obtain:

\[
\frac{h}{T} \approx \frac{1}{3} \left( \frac{T_c}{T} \right)^3 m^3
\]

This implies \( \delta = 3 \).