Lax Pairs for the KdV equation

First we define the Hamiltonian operator :

```
In[146]:= H = (4 D[#, {x, 3}] - 3 (D[u[x] * #, x] + u[x] * D[#, x])) &
```

Out[146]= $4\,\partial_{\{x,3\}}\#1 - 3\,(\partial_x(u[x]\,\#1) + u[x]\,\partial_x\#1)\,\&$

Example for the use of the Hamiltonian :

```
In[129]:= H[x^4]
```

Out[129]= $96\,x - 3\,(8\,x^3\,u[x] + x^4\,u'[x])$

Next, define the Schroedinger operator :

```
In[135]:= L = (-D[#, {x, 2}] + u[x] * #) &
```

Out[135]= $-\partial_{\{x,2\}}\#1 + u[x]\,\#1\,\&$

Example :

```
In[136]:= L[x^2]
```

Out[136]= $-2 + x^2\,u[x]$

Applying the L to the H to an arbitrary function :

```
In[142]:= L@H@f[x]
```

Out[142]= $u[x]\,(-3\,(2\,u[x]\,f'[x] + f[x]\,u'[x]) + 4\,f^{(3)}[x]) + 3\,(5\,u'[x]\,f''[x] + 4\,f'[x]\,u''[x] + 2\,u[x]\,f^{(3)}[x] + f[x]\,u^{(3)}[x]) - 4\,f^{(5)}[x]$

Applying the H to the L and to an arbitrary function :

```
In[143]:= H@L@f[x]
```

Out[143]= $-3\,(u'[x]\,(f[x]\,u[x] - f''[x]) + 2\,u[x]\,(u[x]\,f'[x] + f[x]\,u'[x] - f^{(3)}[x])) + 4\,(3\,u'[x]\,f''[x] + 3\,f'[x]\,u''[x] + u[x]\,f^{(3)}[x] + f[x]\,u^{(3)}[x] - f^{(5)}[x])$

The commutator, [L, H], is the difference between the two operators above

```
In[149]:= L@H@f[x] - H@L@f[x]
```

Out[149]= $3\,(u'[x]\,(f[x]\,u[x] - f''[x]) + 2\,u[x]\,(u[x]\,f'[x] + f[x]\,u'[x] - f^{(3)}[x])) + u[x]\,(-3\,(2\,u[x]\,f'[x] + f[x]\,u'[x]) + 4\,f^{(3)}[x]) + 3\,(5\,u'[x]\,f''[x] + 4\,f'[x]\,u''[x] + 2\,u[x]\,f^{(3)}[x] + f[x]\,u^{(3)}[x]) - 4\,(3\,u'[x]\,f''[x] + 3\,f'[x]\,u''[x] + u[x]\,f^{(3)}[x] + f[x]\,u^{(3)}[x] - f^{(5)}[x]) - 4\,f^{(5)}[x]$

Looks bad, but :

```
In[150]:= % // Expand
```

Out[150]= $6\,f[x]\,u[x]\,u'[x] - f[x]\,u^{(3)}[x]$

which is f[x] times the KdV equation.