Problem set due - Wednesday Oct. 13th, 5pm, Gil's mailbox in Sloan Annex.

Problem set 2 - Green functions

1. Convolution theorem. Consider a differential operator $\mathcal{L}_{\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)}$, which has a Green function $G(x-x',t-t')$ such that:

$$\mathcal{L}_{\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)}G(x-x',t-t') = \delta_{(x-x')}\delta_{(t-t')}. \tag{1}$$

   Show that the Fourier transform of the solution for the non-homogeneous equation:

$$\mathcal{L}_{\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)}\psi(x,t) = f(x,t) \tag{2}$$

   is:

$$\psi_{k,\omega} = \left(\mathcal{L}(ik,-i\omega)\right)^{-1}\cdot f_{k,\omega} \tag{3}$$

2. Diffusion with an expiration date. In many situations we are interested in the diffusion of paritcles or object classes that decay. Examples range from radio-active materials to polpulation dynamics. The PDE that describes such a situation is:

$$\frac{\partial\rho}{\partial t} - D\frac{\partial^2\rho}{\partial x^2} + \lambda\rho = 0 \tag{4}$$

   with λ a constant determining the rate of decay. Find the Green function describing this problem, $G(x,t)$.

3. Schrödinger equation on a lattice. The wave function of a quantum particle in a tight binding lattice (e.g., an electron in a 1d chain of atoms) is described by the following equation:

$$i\frac{\partial\psi_n(t)}{\partial t} = -w\left(\psi_{n+1}(t)+\psi_{n-1}(t)\right) \tag{5}$$

   with w being the hopping strength between neighboring sites, and n is the site index.

   (a) Show that any discrete normalizable function f_n can be written as:

$$f_n = \int_{-\pi}^{\pi}\frac{dk}{2\pi}e^{ikn}f_k. \tag{6}$$

   with the Fourier component $f_k = \sum_n e^{-ikn}f_n$.

   (b) Using Fourier methods prove that the Green function for this problem, which obeys:

$$i\frac{\partial G_n(t)}{\partial t} + w\left(G_{n+1}(t)+G_{n-1}(t)\right) = \delta_{n,0}\delta(t) \tag{7}$$

   (with $\delta_{n,0}$ is a Kronecker δ) is:

$$G_n(t) = i^{|n|+1}\mathcal{J}_{|n|}(2wt) \tag{8}$$

   with $\mathcal{J}_n(t)$ the n'th Bessel function.

   (c) * Plot the probability profile of the particle at time $t = 5, 10, 20, 30, 40$. Plot the location of the wave front as a function of time for this range. What determines the velocity of the front?

   A useful identity for this problem is the generating funtion of the Bessel functions:

$$e^{\frac{z}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n\mathcal{J}_n(z). \tag{9}$$