Physics 129A, Fall 2010
Problem Set 2 Solution

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1. Convolution theorem

Let $\mathcal{F}\{f\}(k), \mathcal{F}\{g\}(k)$ be the Fourier transform of functions $f(x), g(x)$, respectively. Then the convolution of two functions $f(x)$ and $g(x)$ is defined as

$$f * g = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(y)f(x-y)dy. \tag{1}$$

The convolution theorem states that the Fourier transform of $f * g$ is $\mathcal{F}\{f\}\mathcal{F}\{g\}$, *i.e.*

$$\mathcal{F}\{f * g\} = \mathcal{F}\{f\}\mathcal{F}\{g\}, \tag{2}$$

where $\mathcal{F}$ means Fourier transform. This can be shown as follows;

$$\begin{aligned}\int_{-\infty}^{\infty} g(y)f(x-y)dy &= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(y)\int_{-\infty}^{\infty} F(k)e^{ik(x-y)}dkdy \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\left[\int_{-\infty}^{\infty} g(y)e^{-iky}\right]e^{ikx}dk \\ &= \int_{-\infty}^{\infty} F(k)G(k)e^{ikx}dk.\end{aligned} \tag{3}$$

In this problem, we are given,

$$\mathcal{L}_{\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)}\psi(x,t) = f(x,t). \tag{4}$$

Therefore, given $\mathcal{L}_{\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)}G(x-x',t-t') = \delta(x-x')\delta(t-t')$,

$$\psi(x,t) = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dt' G(x-x',t-t')f(x',t'). \tag{5}$$

With slight modification of eq(3), we get

$$\psi(x,t) = \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dw G(k,w)f_{k,w}e^{ikx-iwt}, \tag{6}$$

where $f_{k,w}$ is the Fourier transform of $f(x,t)$.
Let $\psi_{k,w}$ be the Fourier transform of $\psi(x,t)$.

Then,

$$\psi_{k,w} = \int_{-\infty}^{\infty} \frac{dx}{2\pi} \int_{-\infty}^{\infty} \frac{dt}{2\pi} \psi(x,t) e^{-ikx+iwt} \tag{7}$$

$$= \int_{-\infty}^{\infty} \frac{dx}{2\pi} \int_{-\infty}^{\infty} \frac{dt}{2\pi} \int_{-\infty}^{\infty} dk' \int_{-\infty}^{\infty} dw' G(k',w') f_{k',w'} e^{ik'x-iw't} e^{-ikx+iwt} \tag{8}$$

$$= \int_{-\infty}^{\infty} dk' \int_{-\infty}^{\infty} dw' G(k',w') f_{k',w'} \delta(k-k')\delta(w-w') \tag{9}$$

$$= G(k,w) f_{k,w} \tag{10}$$

By the way, from the equation $\mathcal{L}_{\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)} G(x-x', t-t') = \delta(x-x')\delta(t-t')$,

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{dw}{2\pi} e^{ik(x-x')} e^{-iw(t-t')} = \mathcal{L}_{\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{dw}{2\pi} G(k',w') e^{ik(x-x')} e^{-iw(t-t')} \tag{11}$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{dw}{2\pi} G(k',w') \mathcal{L}_{\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)} e^{ik(x-x')} e^{-iw(t-t')} \tag{12}$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{dw}{2\pi} G(k',w') \mathcal{L}_{ik,-iw} e^{ik(x-x')} e^{-iw(t-t')} \tag{13}$$

Therefore, $G(k',w')\mathcal{L}_{ik,-iw} = 1$, or $G(k',w') = (\mathcal{L}_{ik,-iw})^{-1}$
Thus, $\psi_{k,w} = G(k,w) f_{k,w} = (\mathcal{L}_{ik,-iw})^{-1} f_{k,w}$.

This problem can also be done simply by plugging the Fourier transform of $\psi(x,t), f(x,t)$ into eq(4).

2. Diffusion with an expiration date.

Difference between the homogeneous diffusion equation $\left(\frac{\partial \rho}{\partial t} - \frac{\partial^2 \rho}{\partial x^2} = 0\right)$ in class and the diffusion equation here $\left(\frac{\partial \rho}{\partial t} - D\frac{\partial^2 \rho}{\partial x^2} + \lambda\rho = 0\right)$ is the presence of the term $\lambda\rho$ and the coefficient D. The way to get Green's function for this problem is very similar to the case of diffusion equation which has been discussed in detail in class. Therefore, discussion here will be brief, and please refer to the lecture note of October 6th for detail.
Green's function satisfies

$$\frac{\partial G(x,t)}{\partial t} - D\frac{\partial^2 G(x,t)}{\partial x^2} + \lambda G(x,t) = \delta(x)\delta(t). \tag{14}$$

Plugging $G(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{dw}{2\pi} e^{ikx-iwt} G_{k,w}$ into eq(14), we get

$$G_{k,w} = \frac{i}{w + i(Dk^2+\lambda)}. \tag{15}$$

Therefore,

$$G(x,t) = i\int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{dw}{2\pi} \frac{e^{ikx-iwt}}{w + i(Dk^2+\lambda)}. \tag{16}$$

The pole is in lower half of complex plane of w. For non-vanishing solution for $t > 0$, we take contour in bottom half as in class. (λ is real). Then we get,

$$G(x,t) = i\int_{-\infty}^{\infty} \frac{dk}{2\pi}\frac{-2\pi i}{2\pi} e^{ikx} e^{-(Dk^2+\lambda)t} \tag{17}$$

$$= e^{-\lambda t}\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} e^{-Dk^2 t} \tag{18}$$

$$= e^{-\lambda t}\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-Dt\left(k-\frac{ix}{2Dt}\right)^2-\frac{x^2}{4Dt}} \tag{19}$$

$$= e^{-\lambda t}\frac{\sqrt{\pi}}{2\pi\sqrt{Dt}} e^{\frac{-x^2}{4Dt}} \tag{20}$$

$$= \frac{1}{2\sqrt{\pi Dt}} e^{-\lambda t} e^{\frac{-x^2}{4Dt}}, \tag{21}$$

for $t > 0$, where the minus sign of $-2\pi i$ comes from the path of integration of contour integration. For $t < 0$, $G(x,t) = 0$ because the contour in upper half of complex plane of w doesn't contain pole.

3. Schödinger equation on a lattice

(a)

By direct calculation we can show that $\delta_{n,n'} = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{ik(n-n')}$. Therefore, a given discrete normalizable function f_n whose Fourier component f_k is $f_k = \sum_n e^{-ikn} f_n$,

$$\int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{ikn} f_k = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{ikn} \sum_{n'} e^{-ikn'} f_{n'} \tag{22}$$

$$= \sum_{n'} \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{ik(n-n')} f_{n'} \tag{23}$$

$$= \sum_{n'} \delta_{n,n'} f_{n'} \tag{24}$$

$$= f_n. \tag{25}$$

(b)

Let $\tilde{G}_k(\eta)$ be Fourier transform of $G_n(t)$;

$$G_n(t) = \int_{-\infty}^{\infty} \frac{d\eta}{2\pi} e^{-i\eta t} \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{ikn} \tilde{G}_k(\eta). \tag{26}$$

From differential equation,

$$i\frac{\partial G_n(t)}{\partial t} + w\left(G_{n+1}(t) + G_{n-1}(t)\right) = \delta_{n,0}\delta(t), \tag{27}$$

we get,

$$\eta\tilde{G}_k(\eta) + w\left(e^{ik}\tilde{G}_k(\eta) + e^{-ik}\tilde{G}_k(\eta)\right) = 1 \tag{28}$$

$$\rightarrow \quad \tilde{G}_k(\eta) = \frac{1}{\eta + w(e^{ik} + e^{-ik})}. \tag{29}$$

Plugging back eq(29) into eq(26),

$$G_n(t) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{ikn} \int_{-\infty}^{\infty} \frac{d\eta}{2\pi} e^{-i\eta t} \frac{1}{\eta + w(e^{ik} + e^{-ik})} \tag{30}$$

The pole is on the real axis of complex plane of η. We want solution which doesn't vanish for $t > 0$, so we give a pole prescription so that pole is at the bottom half of complex plane. Then contour integral with contour in lower half plane picks up the pole. Therefore, for $t > 0$,

$$G_n(t) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{ikn} \frac{-2\pi i}{2\pi} e^{iwt(e^{ik}+e^{-ik})} \tag{31}$$

$$= -i \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{ikn} e^{iwt(e^{ik}+e^{-ik})}. \tag{32}$$

By the way, if we choose $z = 2wt$ and $t = ie^{ik}$, then from $e^{\frac{z}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} t^n \mathcal{J}_n(z)$, we get

$$\begin{aligned} \sum_{n=-\infty}^{\infty} i^n e^{ikn} \mathcal{J}_n(2wt) &= e^{wt(ie^{ik} - \frac{1}{ie^{ik}})} \\ &= e^{iwt(e^{ik}+e^{-ik})} \end{aligned} \tag{33}$$

Plugging eq(33) into eq(32),

$$G_n(t) = -i \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{ikn} e^{iwt(e^{ik}+e^{-ik})} \tag{34}$$

$$= -i \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{ikn} \sum_{n'=-\infty}^{\infty} i^{n'} e^{ikn'} \mathcal{J}_{n'}(2wt) \tag{35}$$

$$= -i \sum_{n'=-\infty}^{\infty} i^{n'} \mathcal{J}_{n'}(2wt) \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{ik(n+n')} \tag{36}$$

$$= -i \sum_{n'=-\infty}^{\infty} i^{n'} \mathcal{J}_{n'}(2wt) \delta_{n,-n'} \tag{37}$$

$$= -i^{-n+1} \mathcal{J}_{-n}(2wt). \tag{38}$$

Since $\mathcal{J}_{-n}(x) = (-1)^n \mathcal{J}_n(x)$,

$$G_n(t) = -i^{-n+1} \mathcal{J}_{-n}(2wt) \tag{39}$$

$$= -i^{n+1} \mathcal{J}_n(2wt). \tag{40}$$

Therefore,

$$G_n(t) = -i^{|n|+1} \mathcal{J}_{|n|}(2wt). \tag{41}$$

(c)

The probability function $P_n(t)$ is given as

$$P_n(t) = G_n^*(t) G_n(t). \tag{42}$$

Fig(1) and Fig(2) are the plots for probability profile of the particle at time $t = 5, 10, 20, 30, 40$ when $w = 1/2$ and $w = 1$, respectively.
The location of the wave front is set as the farthest local maximum of the wave from the origin in Fig(3). The blue line and the red line are for the case of $w = 1/2$ and $w = 1$, respectively.
As shown in Fig(3), the velocity of the wave front is determined by w. Actually, from the dispersion relation $\eta = -w(e^{ik} + e^{-ik}) = -2w \cos k$, the group velocity v_g of the wave is $v_g = \frac{d\eta}{dk} = 2w \sin k$.

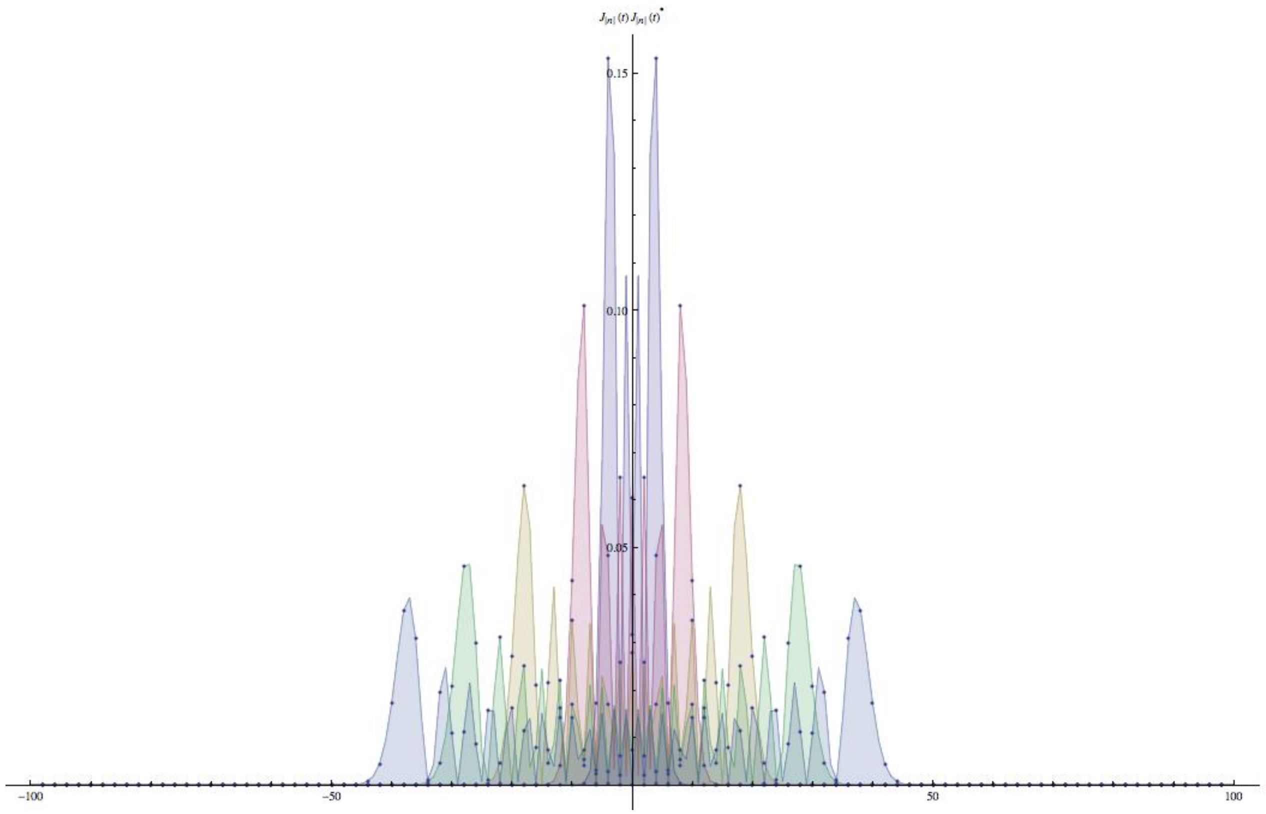

Figure 1: Plot for probability profile of the particle at time $t = 5, 10, 20, 30, 40$ when $w = 1/2$

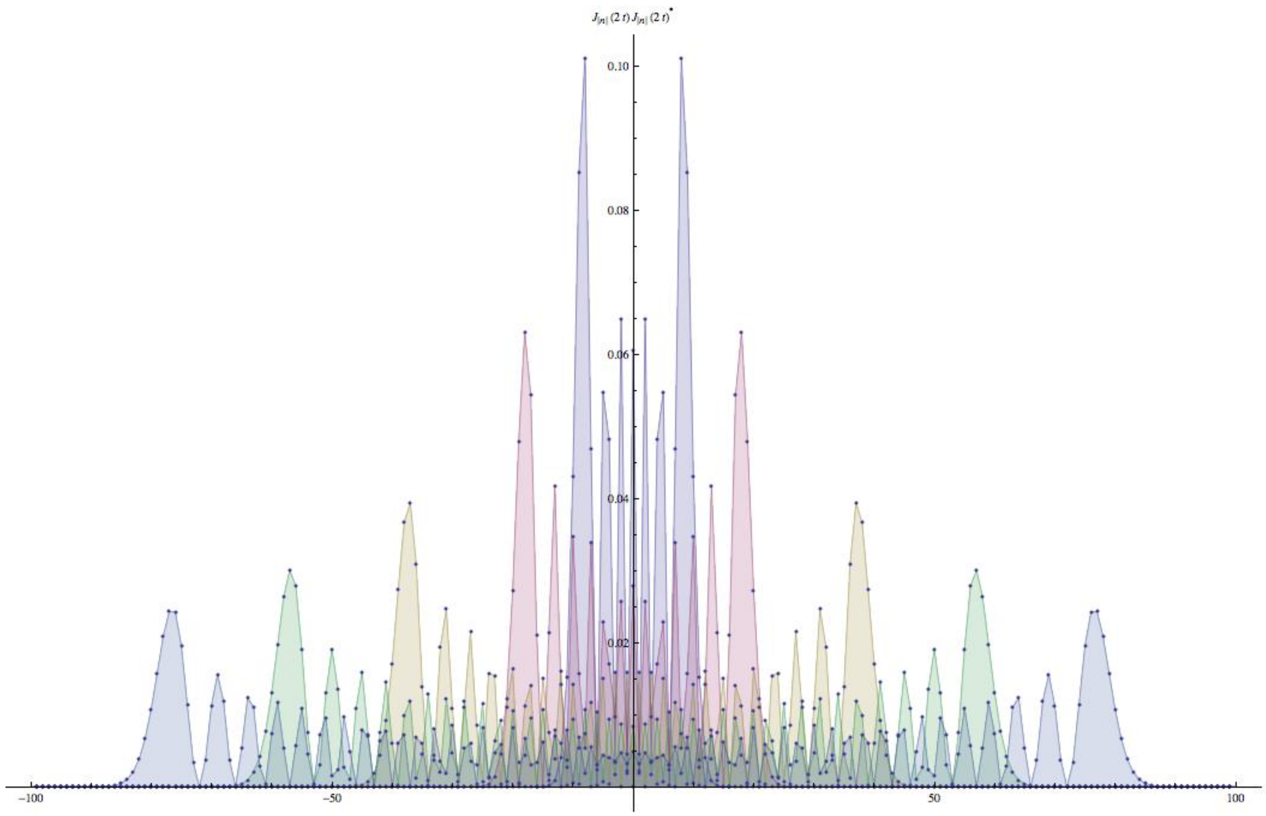

Figure 2: Plot for probability profile of the particle at time $t = 5, 10, 20, 30, 40$ when $w = 1$

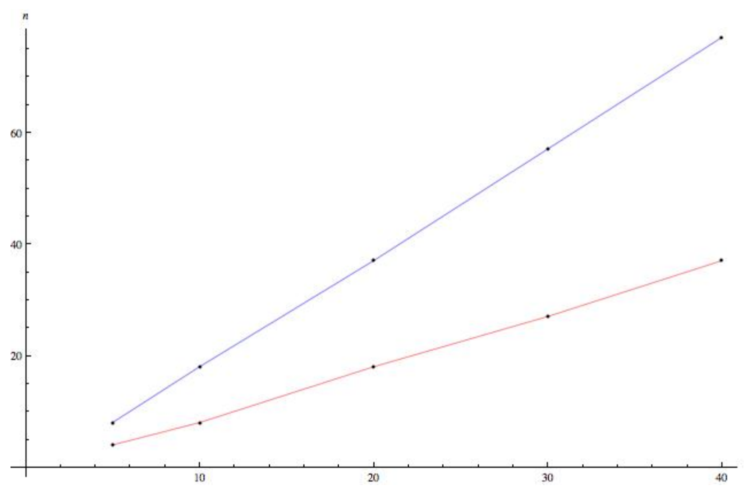

Figure 3: Plot for the location of wave front as a function of time for above time range