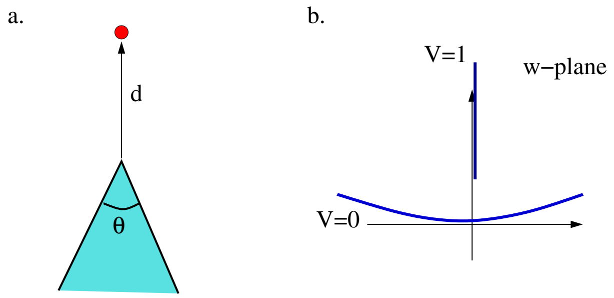

FIG. 1: (a) A charge is placed near a metallic wedge of angle θ. In 1b you are requested to find the electric field when the charge is placed right above the wedge (on the angle bisector of the wedge). (b) In 2 you are asked to find the field between the two metallic lines - the parabola at voltage $V = 0$, and the semi-infinite line on $Im w > 1/4$ with voltage $V = 1$.

Problem set due - Friday Oct. 29th, 5pm

Problem set 4 - Conformal Mappings and the Laplace equation

1. The field of a tip. A well known principle in electrostatics is that the field near sharp metallic objects is strongly enhanced. In this problem we will explore this principle in 2d using conformal mapping.
   (a) A point particle with unit charge is placed near a wedge of angle θ (assume a wedge of infinite sides). What is the electric potential as a function of position (you can define your own parametrization of space) and of the particle's position?
   (b) Consider the case where the particle is placed directly above the wedge at a distance d (see Fig. 1). What is the electric field on the line between the wedge and the particle as a function of distance from the wedge?

2. Needle edge.
   (a) Find a conformal mapping which maps the real line $z = x$ to the parabola $w = x' + iy'$ with $y' = x'^2$.
   (b) The answer to the previous question is not unique. The simplest transformation perhaps maps the strip $0 < Im z < 1/2$ to the region contained between the parabola above and the line $Re w = 0$, $Im w \geq 1/4$. What is this mapping?
   (c) Consider the space between the parabola $w = x' + ix'^2$ and the needle $Re w = 0$, $Im w \geq 1/4$. If a potential difference of $\Delta V = 1$ exists between the parabola and the needle, what is the potential as a function of position?
   (d) What is the electric field as a function of distance from the tip of the needle along the line $Re w = 0$?

3. Given the green function $V = Re\phi(z)$, and the complex function $\phi(z)$, of a point charge (located at z_0) as a function of $z = x + iy$. Prove that a conformal mapping $z = z(w)$ to the function $V = \phi(z(w)) = \tilde{V}(w)$ leaves the charge of the point particle at hand (now at $w_0 = w(z_0)$, with $w(z)$ the inverse function to $z(w)$) unchanged.

4. Möbius mappings and circular images. In class we described the mapping:

$$z(w) = \frac{aw + b}{cw + d} \tag{1}$$

   which maps circles to circles. It is convenient to express the information of this transformation as a matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2}$$

   (a) Show that the inverse transformation to Eq. (1), $w = \frac{a'z+b'}{c'z+d'}$, has coefficients:

$$A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = A^{-1} \tag{3}$$

(b) Show that compounding two Möbius transformations, Eq. (1), and $w(v) = \frac{ev+f}{gv+h}$, represented by matrix B, is represented by the product:

$$C = A \cdot B \tag{4}$$

with the dot representing matrix product.

(c) A point particle with unit charge is placed at $w_0 = x_0$ $(Imw_0 = 0)$ near a (grounded) metallic sphere, whose perimeter is described by the curve $|w| = 1$. What is the potential of the point charge?
Hint: Use the Möbius mapping to map between the unit sphere and the line $Imz = 0$.