Physics 129A, Fall 2010
Problem Set 4 Solution

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1. The field of a tip

(a)

We put a metallic wedge of angle θ and take conformal mapping $z = w^{\frac{\pi}{2\pi-\theta}}$ as in figure (1). The positive real axis in w plane is mapped to the positive real axis in z plane, and the another edge of the metallic wedge in w plane is mapped into the negative real axis of z plane. We put the branch cut at the positive real axis of w plane. The point w_0 in w plane outside of the wedge is mapped to z_0 which is at the 'outside' of the region $Im(z) < 0$. Therefore we can use the formula from the lecture;

$$\phi(z) = -\frac{1}{2\pi}\ln\left(\frac{z - z_0}{z - \bar{z_0}}\right) \tag{1}$$

Therefore,

$$\tilde{\phi}(w) = -\frac{1}{2\pi}\ln\left(\frac{w^{\frac{\pi}{2\pi-\theta}} - w_0^{\frac{\pi}{2\pi-\theta}}}{w^{\frac{\pi}{2\pi-\theta}} - \bar{w}_0^{\frac{\pi}{2\pi-\theta}}}\right), \tag{2}$$

so the electric potential is

$$V(w) = Re[\tilde{\phi}(w)] \tag{3}$$

$$= -\frac{1}{2\pi}\ln\left(\frac{\left|w^{\frac{\pi}{2\pi-\theta}} - w_0^{\frac{\pi}{2\pi-\theta}}\right|}{\left|w^{\frac{\pi}{2\pi-\theta}} - \bar{w_0}^{\frac{\pi}{2\pi-\theta}}\right|}\right). \tag{4}$$

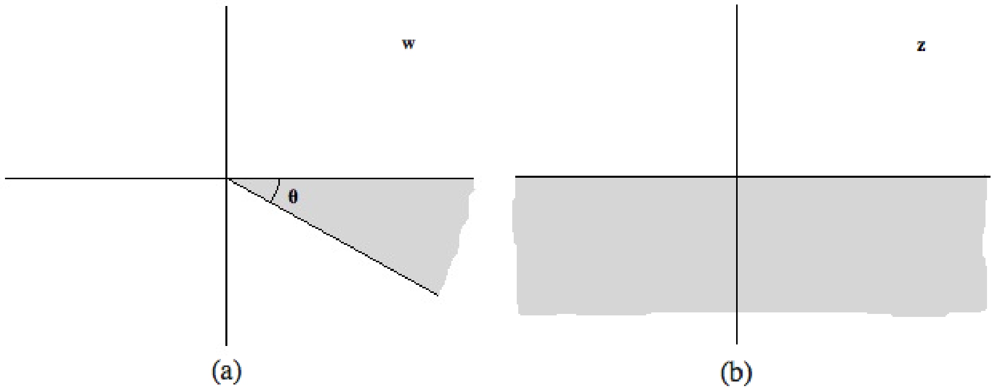

Figure 1: (a) The original problem, (b) The configuration which we know what the electric potential is

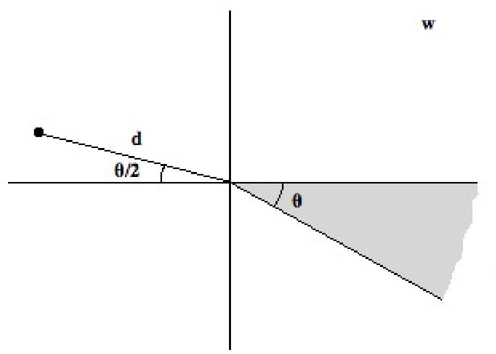

Figure 2: The configuration of Problem 1.(b)

(b)

Let $w_0 = de^{i\frac{2\pi-\theta}{2}}$, and $w = le^{i\frac{2\pi-\theta}{2}}$. By using previous result, for $l < d$

$$\begin{aligned} V(w) &= Re[\tilde{\phi}(w)] \qquad (5) \\ &= -Re\left[\frac{1}{2\pi}\ln\left(\frac{l^{\frac{\pi}{2\pi-\theta}}e^{i\pi/2} - d^{\frac{\pi}{2\pi-\theta}}e^{i\pi/2}}{l^{\frac{\pi}{2\pi-\theta}}e^{i\pi/2} - d^{\frac{\pi}{2\pi-\theta}}e^{i\pi/2}}\right)\right] \qquad (6) \\ &= -Re\left[\frac{1}{2\pi}\ln\left(\frac{l^{\frac{\pi}{2\pi-\theta}} - d^{\frac{\pi}{2\pi-\theta}}}{l^{\frac{\pi}{2\pi-\theta}} + d^{\frac{\pi}{2\pi-\theta}}}\right)\right] \qquad (7) \\ &= -\frac{1}{2\pi}\ln\left(\frac{d^{\frac{\pi}{2\pi-\theta}} - l^{\frac{\pi}{2\pi-\theta}}}{l^{\frac{\pi}{2\pi-\theta}} + d^{\frac{\pi}{2\pi-\theta}}}\right), \qquad (8) \end{aligned}$$

where the condition $l < d$ is used from eq(7) to eq(8).
Due to symmetry, electric field has non-vanishing component only in the radial direction l. Thus,

by evaluating $E_l = -\frac{dV}{dl}$, we get

$$E_l = -\frac{1}{2\pi - \theta} \frac{d^{\frac{\pi}{2\pi-\theta}} l^{\frac{\theta-\pi}{2\pi-\theta}}}{d^{\frac{2\pi}{2\pi-\theta}} - l^{\frac{2\pi}{2\pi-\theta}}}. \tag{9}$$

2. Needle edge

(a)

Conformal mapping $w = z + iz^2$ satisfies the condition, because for $z = x$, $w = x + ix^2$. However, conformal mapping $w = -z + iz^2$ also satisfies the condition, because for $z = x$, $w = -x + ix^2 = x' + iy'$ and $y' = x'^2$.

(b)

We want the conformal mapping which maps the strip $0 < Im[z] < 1/2$ to the region contained between the parabola and the line $Re[w] = 0, Im[w] \leq 1/4$. This condition picks $w = z + iz^2$. This maps the line $z = x + i/2$ in z plane to the needle in w plane, and the real axis of z plane to the parabola in w plane.

(c)

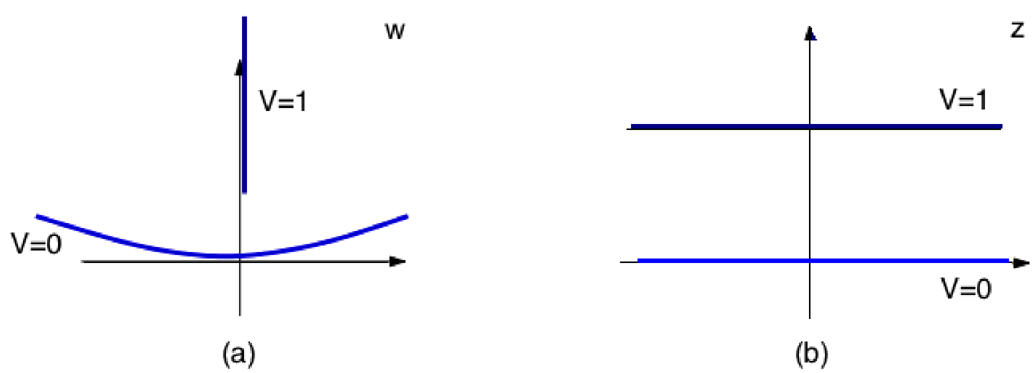

Figure 3: (a) The original problem, (b) The configuration which we know what the electric potential is

In figure(3.(b)), the electric potential is $2y$ where $z = x + iy$. If we set $\phi(z) = -2iz$, then the electric field is given as $Re[\phi(z)]$. From $w = z + iz^2$, we get $z = -i(iw + 1/4)^{1/2} + i/2$, and take the branch cut along the needle, *i.e.* $-3\pi/2 < Arg[w - i/4] \leq \pi/2$. There is actually sign ambiguity in the expression of z, but this one gives correct map in our setup. Actually, if we choose cut as above, $z = -i(iw + 1/4)^{1/2} + i/2$ maps the needle to $z = x + i/2$ with positive x, because of $-3\pi/2 < Arg[w - i/4]$, not $-3\pi/2 \leq Arg[w - i/4]$. However it is okay for our purpose. Then, the electric field is

$$V(w) = Re[\tilde{\phi}(w)] \tag{10}$$

$$= Re[-2(iw + 1/4)^{1/2} + 1] \tag{11}$$

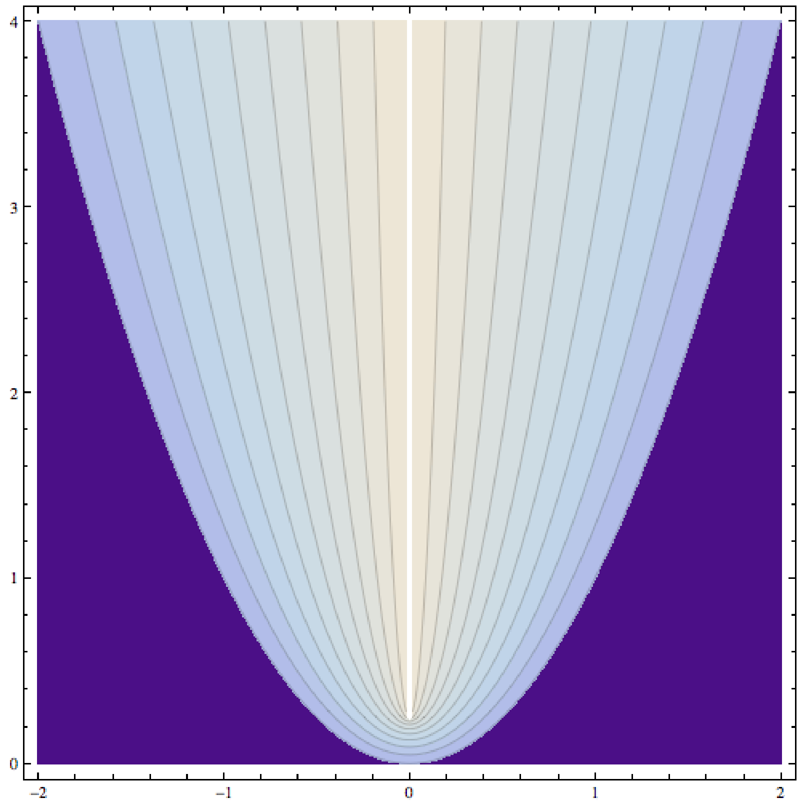

Figure 4: Equipotential lines for the electric potential $V(w)$

(d)

By symmetry along the $Re[w]$, only y component of the electric field is nonvanishing. Let $w = iy$ with $y \in [0, 1/4)$. Then, the potential is

$$V(w) = Re[-2(1/4 - y)^{1/2} + 1] \tag{12}$$
$$= -2(1/4 - y)^{1/2} + 1. \tag{13}$$

Thus, the y-component of the electric field is

$$E_y = -\frac{dV}{dy} \tag{14}$$
$$= -\frac{1}{\sqrt{1/4 - y}}. \tag{15}$$

Since line $Re[w] = 0$ with $Im[w] \geq 1/4$ has same value of potential $V = 1$, electric field for $Re[w] = 0$ with $Im[w] \geq 1/4$ is zero.

3.

We want to check that $\nabla^2 V = q\delta(x-x_0)\delta(y-y_0)$ transformed to same form under conformal transformation where $V = Re[\phi(z)]$. Let $z = x+iy$ and $w = x'+iy'$. From the lecture,

$$\begin{aligned} \nabla^2 Re[\phi(z)] &= 4\frac{\partial^2}{\partial z \partial \bar{z}} Re[\phi(z)] \\ &= 4\frac{\partial w}{\partial z}\frac{\partial \bar{w}}{\partial \bar{z}}\frac{\partial^2}{\partial w \partial \bar{w}}\tilde{V}(w), \end{aligned} \tag{16}$$

where $\tilde{V}(w) = Re[\phi(z(w))]$.
From $z = x+iy$ and $w = x'+iy'$, we can show by direct calculation that

$$\begin{aligned} \frac{\partial w}{\partial z}\frac{\partial \bar{w}}{\partial \bar{z}} &= \left(\frac{\partial x'}{\partial x} + i\frac{\partial y'}{\partial x}\right)\left(\frac{\partial x'}{\partial x} - i\frac{\partial y'}{\partial x}\right) \\ &= \frac{\partial x'}{\partial x}\frac{\partial y'}{\partial y} - \frac{\partial y'}{\partial x}\frac{\partial x'}{\partial y}, \end{aligned} \tag{17}$$

where Cauchy-Riemann condition $\left(\frac{\partial x'}{\partial x} = \frac{\partial y'}{\partial y}, \frac{\partial y'}{\partial x} = -\frac{\partial x'}{\partial y}\right)$ was used at each steps.
We also want to know how delta function is transformed by using the properties of delta function under changes of variables, $\delta(\mathbf{f}(\mathbf{x})) = \frac{\delta(\mathbf{x}-\mathbf{x_0})}{|\det \nabla_{\mathbf{x}}\mathbf{f}(\mathbf{x})|}$ where bold character denotes vector quantities. Thus,

$$\delta(x'(x,y) - x'_0(x_0,y_0))\delta(y'(x,y) - y'_0(x_0,y_0)) = \frac{\delta(x-x_0)\delta(y-y_0)}{\left|\det\begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial y'}{\partial x} \\ \frac{\partial x'}{\partial y} & \frac{\partial y'}{\partial y}\end{pmatrix}\right|} \tag{18}$$

Therefore,

$$\delta(x-x_0)\delta(y-y_0) = \left(\frac{\partial x'}{\partial x}\frac{\partial y'}{\partial y} - \frac{\partial y'}{\partial x}\frac{\partial x'}{\partial y}\right)\delta(x'(x,y) - x'_0(x_0,y_0))\delta(y'(x,y) - y'_0(x_0,y_0)) \tag{19}$$

From eq(16, 17, 19), and $4\frac{\partial^2}{\partial z \partial \bar{z}} Re[\phi(z)] = q\delta(x-x_0)\delta(y-y_0)$,

$$\begin{aligned} \left(\frac{\partial x'}{\partial x}\frac{\partial y'}{\partial y} - \frac{\partial y'}{\partial x}\frac{\partial x'}{\partial y}\right)\delta(x'(x,y) - x'_0(x_0,y_0))\delta(y'(x,y) - y'_0(x_0,y_0)) &= \delta(x-x_0)\delta(y-y_0) && (20) \\ &= \nabla^2 Re[\phi(z)] && (21) \\ &= 4\left(\frac{\partial x'}{\partial x}\frac{\partial y'}{\partial y} - \frac{\partial y'}{\partial x}\frac{\partial x'}{\partial y}\right)\frac{\partial^2}{\partial w \partial \bar{w}}\tilde{V}(w). && (22) \end{aligned}$$

Therefore,

$$4\frac{\partial^2}{\partial w \partial \bar{w}}\tilde{V}(w) = q\delta(x' - x'_0))\delta(y' - y'_0). \tag{23}$$

Hence, q doesn't change.

It can be done in another way. From the expression $\oint d\overrightarrow{r} \cdot \nabla V = q$ where $V = Re[\phi(z)]$ by changing variables to complex coordinates,

$$q = \frac{1}{2}\oint dz\frac{\partial \phi(z)}{\partial z} + d\bar{z}\frac{\partial \bar{\phi}(\bar{z})}{\partial \bar{z}}. \tag{24}$$

Under conformal mapping, above expression becomes

$$\frac{1}{2}\oint dw \frac{\partial \phi(z(w))}{\partial w} + d\bar{w}\frac{\partial \bar{\phi}(\bar{z}(\bar{w}))}{\partial \bar{w}} = \oint d\vec{r}' \cdot \nabla' \tilde{V}, \tag{25}$$

where the contour enclosing the pole in z plane is mapped to the contour enclosing the pole in w plane.
Therefore q doesn't change.

4. Möbius mappings and circular images.

(a)

From $z(w) = \frac{aw+b}{cw+d}$, $w = \frac{-dz+b}{cz-a}$. Therefore, $a' = -d, b' = b, c' = c, d' = -a$. Since $ad - bc \neq 0$, if we multiply numerator and denominator, it is actually inverse of A.

(b)

$$z(w(u)) = \frac{\frac{aeu+af}{gu+h} + b}{\frac{ceu+cf}{gu+h} + d} = \frac{aeu + af + bgu + bh}{ceu + cf + dgu + dh} = \frac{(ae+bg)u + af + bh}{(ce+dg)u + cf + dh}. \tag{26}$$

Thus, $C = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = A \cdot B$

(c)

From the lecture, $z = \frac{w-w_1}{w-w_2}\frac{w_3-w_2}{w_3-w_1}$ maps w_1, w_2, w_3 on the circle in w plane to $0, \infty, 1$ on the real axis of z plane. Here we choose $w_1 = 1, w_2 = -1, w_3 = i$. Then $z = i\frac{w-1}{w+1}$. Inside of the circle is mapped into the region $Im[z] < 0$. The position $w_0 = x_0$ of the point particle with unit charge is mapped to $z_0 = i\frac{x_0-1}{x_0+1}$, which is pure imaginary and is above the real axis in z plane. Fig(5).
We know the electric potential of configuration fig(5.(b)). Therefore,

$$\phi(z(w)) = -\frac{1}{2\pi}\ln\left(\frac{z-z_0}{z-\bar{z}_0}\right) = -\frac{1}{2\pi}\ln\left(\frac{i\frac{w-1}{w+1} - i\frac{x_0-1}{x_0+1}}{i\frac{w-1}{w+1} + i\frac{x_0-1}{x_0+1}}\right) \tag{27}$$

$$= -\frac{1}{2\pi}\left[\ln(w-x_0) - \ln(x_0 w - 1)\right] \tag{28}$$

Hence, the potential is

$$V = Re[\tilde{\phi}(w)] = -\frac{1}{2\pi}\left[\ln(|w-x_0|) - \ln(|x_0 w - 1|)\right] \tag{29}$$

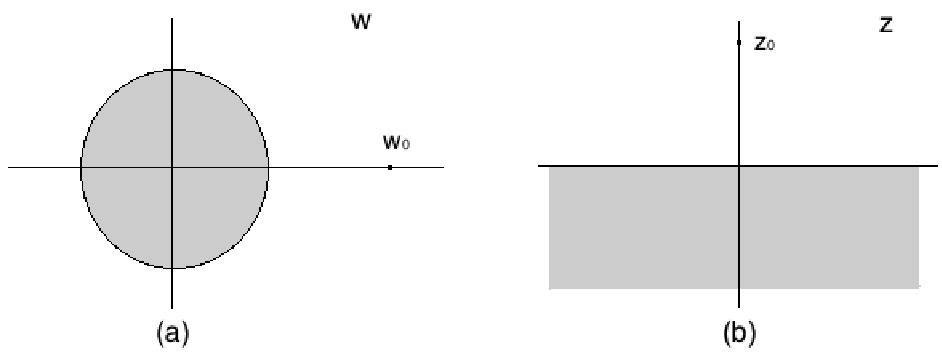

Figure 5: (a) The original problem, (b) The configuration which we know what the electric potential is