Problem set due - Thursday Nov. 4th, 5pm

# Problem set 5 - Solitons etc.

1. Bäcklund transformations. In the theory of analytic functions we ran into the Cauchy-Riemann relations. One way of stating them is that if $u$ is a real solution of $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})u = 0$ and if:

$$u_x = v_y, \; u_y = -v_x \tag{1}$$

then $v$ is also a solution of the Laplace 2D equation $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})v = 0$. This is the simplest example of a Bäcklund transformation.

Here's a more interesting example. Consider the equation:

$$u_{xy} = e^u \tag{2}$$

   (a) Show that if $v$ obeys:

$$v_x = u_x + 2ae^{\frac{u+v}{2}}, \; and \; - v_y = u_y + \frac{1}{a}e^{\frac{u-v}{2}} \tag{3}$$

   then $v$ is a solution of the equation:

$$v_{xy} = 0 \tag{4}$$

   (b) Use the Bäcklund transform, Eqs. (3) and (4), to derive two independent solutions of (2). Hint: use simple solutions for $V_{xy} = 0$, and solve for $u$ from the Bäcklund transform.

2. Traveling wave solution for the sine-Gordon equation. The sine-Gordon equation is:

$$u_{xx} - u_{tt} = \sin u. \tag{5}$$

Find a traveling wave solution for this equation, which is a function of the velocity of the wave, $v$, and which can be assumed to have a vanishing slope at $|x - vt| \to \infty$. Note that the simplest solution will only be zero on one side of the $x$-axis. The answer is usually referred to as an instanton.

3. Inverse scattering for a double spike. The initial conditions for the KdV like equation:

$$v_t + 6vv_x + v_{xxx} = 0 \tag{6}$$

are $v(x, t = 0) = V\,(\delta(x - 1) + \delta(x + 1))$.

   (a) Find a transformation on $u = f(v)$ such that $u$ evolves under the KdV equation: $u_t - 6uu_x + u_{xxx} = 0$.

   (b) What solitons emerge and move to the right due to the initial conditions?