Physics 129A, Fall 2010
Problem set 5 Solution

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November 17, 2010

Problem 1. Bäcklund transformations

(a)

$$v_x = u_x + 2ae^{(u+v)/2} \tag{1}$$

applying $\frac{\partial}{\partial y}$ to EQ(1), then we get:

$$v_{xy} = u_{xy} + 2ae^{(u+v)/2}\frac{1}{2}(u_y + v_y) \tag{2}$$

$$-v_y = u_y + \frac{1}{a}e^{(u-v)/2} \tag{3}$$

applying $\frac{\partial}{\partial x}$ to EQ(3), then we get:

$$-v_{xy} = u_{xy} + \frac{1}{a}e^{(u-v)/2}\frac{1}{2}(u_x - v_x) \tag{4}$$

by EQ(2)(3), we get:

$$v_{xy} = u_{xy} + ae^{(u+v)/2}(\frac{-1}{a})e^{(u-v)/2} = u_{xy} - e^u = 0 \tag{5}$$

(b)
One of the simplest solution for $v_{xy} = 0$ is $v =$ constant.
if $v =$ constant, then by EQ(1)(3)

$$0 = u_x + 2ae^{(u+v)/2} \tag{6}$$

$$0 = u_y + \frac{1}{a}e^{(u-v)/2} \tag{7}$$

since v is constant, so let's assume $c_3 = ae^{v/2}$

$$-u_x = 2ae^{(u+v)/2} = 2c_3e^{u/2} \tag{8}$$

$$-u_y = \frac{1}{a}e^{(u-v)/2} = \frac{1}{c_3}e^{u/2} \tag{9}$$

$$\frac{EQ(8)}{EQ(9)} = \frac{u_x}{u_y} = 2c_3^2 \tag{10}$$

by EQ(10), we get:

$$u = F(2c_3^2 x + y) \tag{11}$$

F is the function that we need to determine later, Let $\eta = 2c_3^2 x + y$.
by EQ(8)(11), we get:

$$-c_3 \frac{dF}{d\eta} = e^{F/2} \tag{12}$$

$$F = -2 * ln(\frac{\eta}{2c_3} + c_4) \tag{13}$$

$$u = F(2c_3^2 x + y) = -2 * ln(c_3 x + \frac{1}{2c_3} y + c_4) \tag{14}$$

c_4 is constant.

Problem 2. Sine-Gordon equation

for traveling solution:

$$u = f(x - vt) \tag{15}$$

let $\eta = x - vt$,the sine-Gordon equation $u_{xx} - u_{tt} = sin(u)$ become

$$(1 - v^2) f'' = sin(f) \tag{16}$$

Let

$$f(\eta) = c_1 arctan(g(\eta)) \tag{17}$$

$$g(\eta) = tan(f(\eta)/c_1) \tag{18}$$

$$sin(f/c_1) = \frac{g}{\sqrt{1 + g^2}} \tag{19}$$

$$cos(f/c_1) = \frac{1}{\sqrt{1 + g^2}} \tag{20}$$

$$\frac{d^2 f}{d\eta^2} = c_1 \frac{g'' + g^2 g'' - 2g(g')^2}{(1 + g^2)^2} = c_1 \frac{g'' + g^2 g'' - 2g(g')^2}{(\sqrt{1 + g^2})^4} \tag{21}$$

$$sin(f) = 2sin(f/2)cos(f/2) = 4sin(f/4)cos(f/4)(2cos^2(f/4) - 1) \tag{22}$$

be the denominator of EQ(21), and EQ(19)(20)(22), it suggest that :

$$c_1 = 4 \tag{23}$$

putting Eq(22)(21) back into Eq(16), we get:

$$(1 - v^2) * [g'' + g^2 g'' - 2g(g')^2] = g * [1 - g^2] \tag{24}$$

Let:

$$g(\eta) = e^{h(\eta)} \tag{25}$$

substitute Eq(25)into Eq(24), we get:

$$[(1-v^2)(h''+(h')^2)-1]e^{h}+[(1-v^2)(h''-(h')^2)+1]e^{3h}=0 \tag{26}$$

Eq(26)suggest that we should find a solution of h, such that it satisfied:

$$h''+(h')^2=\frac{1}{1-v^2} \tag{27}$$

$$h''-(h')^2=\frac{-1}{1-v^2} \tag{28}$$

Eq(27)(28) suggeetst that $h''=0$, so

$$h=c_3\eta+c_2 \tag{29}$$

substitute Eq(29) into Eq(28), we get:

$$h=\frac{1}{\sqrt{1-v^2}}\eta+c_2 \tag{30}$$

by Eq(30)Eq(25):

$$g=e^{\frac{1}{\sqrt{1-v^2}}\eta+c_2} \tag{31}$$

$$u=f(x-vt)=4*arctan[e^{\frac{1}{\sqrt{1-v^2}}(x-vt)+c_2}] \tag{32}$$

Problem 3. Inverse scattering for a double spike

(a)
$u=-v$

(b)
Schrodinger Equation is :

$$-\psi_{xx}-V[\delta(x-1)+\delta(x+1)]\psi=\lambda\psi \tag{33}$$

For bound state,the energy $\lambda<0$.
We have symmetric bound state wave function:

$$\psi_s=\begin{cases} e^{-\sqrt{|\lambda_s|}(x-1)} & \text{if } 1<x \\ \frac{1}{2cosh(\sqrt{|\lambda_s|})}[e^{-\sqrt{|\lambda_s|}x}+e^{\sqrt{|\lambda_s|}x}] & \text{if } -1<x<1 \\ e^{\sqrt{|\lambda_s|}(x+1)} & \text{if } x<-1 \end{cases} \tag{34}$$

with symmetric bound state energy λ_s, such that it satisfied:

$$V=\frac{2\sqrt{|\lambda_s|}e^{\sqrt{|\lambda_s|}}}{e^{-\sqrt{|\lambda_s|}}+e^{\sqrt{|\lambda_s|}}} \tag{35}$$

We also have antisymmetric bound state wave function:

$$\psi_a = \begin{cases} e^{-\sqrt{|\lambda_a|}(x-1)} & \text{if } 1 < x \\ \frac{1}{2sinh(\sqrt{|\lambda_a|})}[e^{\sqrt{|\lambda_a|}x} - e^{-\sqrt{|\lambda_a|}x}] & \text{if } -1 < x < 1 \\ -e^{\sqrt{|\lambda_a|}(x+1)} & \text{if } x < -1 \end{cases} \tag{36}$$

with antisymmetric bound state energy λ_a, such that it satisfied:

$$V = \frac{2\sqrt{|\lambda_a|}e^{\sqrt{|\lambda_a|}}}{e^{\sqrt{|\lambda_a|}} - e^{-\sqrt{|\lambda_a|}}} \tag{37}$$

from Fig(1), we could find that $\lambda_s < \lambda_a < 0$, by lecture notes, we know that if you have bound state energy $\lambda_s < \lambda_a < 0$, then the long-time solution of soliton in KdV equation $u_t - 6uu_x + u_{xxx} = 0$ is:

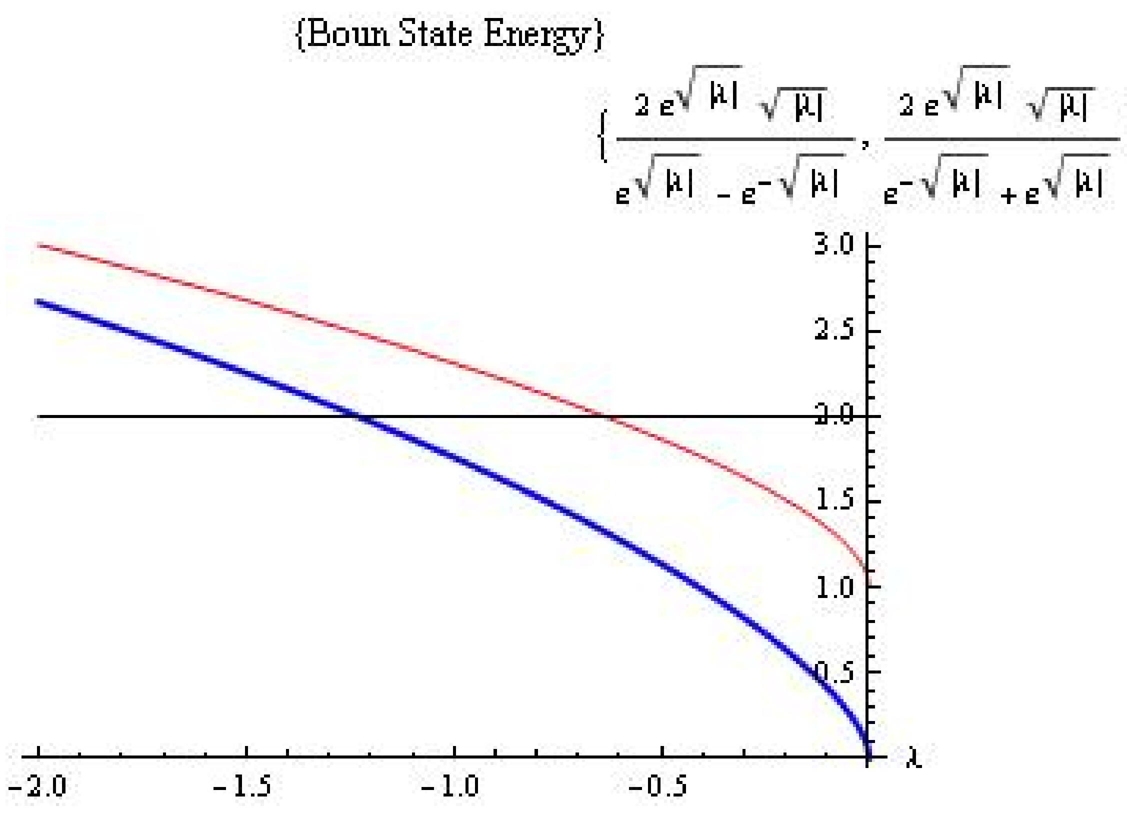
$$u(x,t) = -\frac{2|\lambda_s|}{cosh^2(\sqrt{|\lambda_s|}(x - x_s - 4|\lambda_s|t))} - \frac{2|\lambda_a|}{cosh^2(\sqrt{|\lambda_a|}(x - x_a - 4|\lambda_a|t))} \tag{38}$$

with initial position of soliton

$$x_s = \frac{1}{2\sqrt{|\lambda_s|}}ln(\frac{b_s(0)}{N_s 2\sqrt{|\lambda_s|}}) \tag{39}$$

$$x_a = \frac{1}{2\sqrt{|\lambda_a|}}ln(\frac{b_a(0)}{N_a 2\sqrt{|\lambda_a|}}(\frac{\sqrt{|\lambda_a|} - \sqrt{|\lambda_s|}}{\sqrt{|\lambda_a|} + \sqrt{|\lambda_s|}})^2) \tag{40}$$

From Fig(1), we could also find that the condition of V such that we have two solitons solution is that $V > 1$



Fig(1), the horizontal black line is the value of V, the red curve above is the curve of function

$$\frac{2\sqrt{|\lambda|}e^{\sqrt{|\lambda|}}}{e^{\sqrt{|\lambda|}}-e^{-\sqrt{|\lambda|}}} \text{ v.s. } \lambda \tag{41}$$

the blue curve below is the curve of function

$$\frac{2\sqrt{|\lambda|}e^{\sqrt{|\lambda|}}}{e^{\sqrt{|\lambda|}}+e^{-\sqrt{|\lambda|}}} \text{ v.s. } \lambda \tag{42}$$

the intersection of black line and blue line is the solution of λ_s such that it satisfied:

$$V=\frac{2\sqrt{|\lambda_s|}e^{\sqrt{|\lambda_s|}}}{e^{-\sqrt{|\lambda_s|}}+e^{\sqrt{|\lambda_s|}}} \tag{43}$$

the intersection of red line and black line is the solution of λ_a such that it satisfied:

$$V=\frac{2\sqrt{|\lambda_a|}e^{\sqrt{|\lambda_a|}}}{e^{\sqrt{|\lambda_a|}}-e^{-\sqrt{|\lambda_a|}}} \tag{44}$$