

Physics 129A, Fall 2010  
Midterm Exam Solution

November 29, 2010

**Problem 1. Dissipative string**

(a)

$$\left[\frac{\partial^2}{\partial t^2} + 2\lambda\frac{\partial}{\partial t} + \lambda^2 - \frac{\partial^2}{\partial x^2}\right]G(x, t) = \delta(t)\delta(x) \quad (1)$$

Use Fourier transformation:

$$G(x, t) = \int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} e^{i(kx - \omega t)} \tilde{G}(\omega, k) \quad (2)$$

$$[-\omega^2 + 2\lambda(-i\omega) + \lambda^2 + k^2]\tilde{G}(\omega, k) = 1 \quad (3)$$

$$\tilde{G}(\omega, k) = \frac{1}{(\lambda - i\omega)^2 + k^2} \quad (4)$$

$$G(x, t) = \int \frac{dk}{2\pi} \int \frac{d\omega}{2\pi} \frac{e^{i(kx - \omega t)}}{(\lambda - i\omega)^2 + k^2} \quad (5)$$

$$= (-i)\theta(t) \int \frac{dk}{2\pi} \left[ \frac{e^{ik(x-t)}}{-2ki} + \frac{e^{ik(x+t)}}{2ki} \right] e^{-t\lambda} \quad (6)$$

$$= \frac{\theta(t)}{4\pi} e^{-t\lambda} \int dk \frac{1}{k} [e^{ik(x-t)} - e^{ik(x+t)}] \quad (7)$$

$$\frac{dG}{dx} = \frac{\theta(t)}{4\pi} e^{-t\lambda} \int dk (e^{ik(x-t)} - e^{ik(x+t)}) \quad (8)$$

$$= \frac{\theta(t)}{4\pi} e^{-t\lambda} (\delta(x-t) - \delta(x+t)) \quad (9)$$

so, finally we get Green function:

$$G(x, t) = \frac{1}{2}\theta(t)e^{-t\lambda}[\theta(x+t) - \theta(x-t)] \quad (10)$$

(b)

$$\psi(x, t) = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dt' G(x - x', t - t') \delta(x') \cos(\omega t') \quad (11)$$

$$= \int_{-\infty}^{\infty} dt' G(x, t - t') \cos(\omega t') \quad (12)$$

$$= \int_{-\infty}^{\infty} dt' \cos(\omega t') \frac{1}{2} \theta(t - t') e^{-\lambda(t-t')} [\theta(x + t - t') - \theta(x - t + t')] \quad (13)$$

Let  $t - t' = T$

$$\psi(x, t) = \int_{-\infty}^{\infty} dT \cos(\omega(t - T)) \frac{1}{2} \theta(T) e^{-\lambda T} [\theta(x + T) - \theta(x - T)] \quad (14)$$

$$= \int_0^{\infty} dT \cos(\omega(t - T)) \frac{1}{2} e^{-\lambda T} [\theta(x + T) - \theta(x - T)] \quad (15)$$

$$\theta(x + T) - \theta(x - T) = 1 \text{ if } |x| < T$$

$$\theta(x + T) - \theta(x - T) = 0 \text{ if } |x| > T$$

$$\psi(x, t) = \int_{|x|}^{\infty} dT \cos(\omega(t - T)) \frac{1}{2} e^{-\lambda T} \quad (16)$$

$$= \frac{1}{2} \frac{e^{-\lambda|x|} [\lambda \cos(\omega(|x| - t)) - \omega \sin(\omega(|x| - t))]}{\omega^2 + \lambda^2} \quad (17)$$

## Problem 2. Off-center cylindrical capacitor

Consider conformal mapping,

$$z = \frac{w - w_0}{\bar{w}_0 w - 1}, \quad (18)$$

where  $w_0$  is arbitrary complex number.

## Solution for problem 2

(33 points)

(a)

(12 points for correct proof; 8 points for loss of generality; 0 point for wrong proof )

Since  $|w| = 1$ ,

$$|w - w_0| = |\bar{w}| |w - w_0| = |1 - \bar{w} w_0| = |\bar{w}_0 w - 1|. \quad (19)$$

Therefore,  $|z| = \left| \frac{w - w_0}{\bar{w}_0 w - 1} \right| = 1$ .

(b)

(21 points for correct solution; 13 points for finding correct mapping, 5 points for correct form of potential, 3 points for other parts of the problem including correct calculation.)

By using conformal mapping (18), we know that the unit circle centered at the origin is mapped to itself. So we want to find  $w_0$  such that the circle of cylinder 2 is mapped to the circle centered at the origin. Without loss of generality, we can choose the initial configuration as  $|w| = 1$ ,  $|w - 2/5| = 2/5$ . By the symmetry of the new and the original configurations with respect to the real axis, real axis will be mapped to the real axis, and this condition is satisfied when we choose  $w_0$  as real number. As stated above we want to map these to  $|z| = 1$ ,  $|z| = R$ .

Since conformal mapping (18) maps real axis to real axis, two intersection points of the real line and  $|w - 2/5| = 2/5$ , i.e. 0 and 4/5 will be mapped to two intersection points of the real line and  $|z| = R$ , i.e.  $R$  and  $-R$ . Therefore, without specifying which maps to which, we have

$$\frac{0 - w_0}{0 - 1} = -\frac{4/5 - w_0}{w_0 4/5 - 1}. \quad (20)$$

From eq(20), we get  $w_0^2 - \frac{5}{2}w_0 + 1 = 0$ , and its solutions are  $w_0 = 1/2, 2$ , which means the radius of the circle centered at the origin is  $R = 1/2, 2$ . This means that 0, 4/5 are mapped to  $R, -R$ , respectively. You can actually check that  $|w - 2/5| = 2/5$  is mapped to  $|z| = 1/2$  or 2. We want to map the inner circle to the inner circle, so we choose  $w_0 = 1/2$ .

General form of electric potential in the mapped configuration in z-plane is given as  $\phi(z) = a \ln z + b$ .

The conditions in the problem give,  $\phi(z = 1) = a \ln 1 + b = 0$ , so  $b = 0$ ; and  $\phi(1/2) = a \ln(1/2) = 1$ , so  $a = 1/\ln(1/2)$ .

Therefore, the electric potential in the original configuration is

$$V = |\phi(w)| = \frac{1}{\ln(1/2)} \ln \left| \frac{2w - 1}{w - 2} \right|. \quad (21)$$

### Problem 3. Moving frame Schrödinger equation

$$\left( i \frac{\partial}{\partial t} + \frac{1}{2m} \frac{\partial^2}{\partial k^2} - v \frac{\partial}{\partial k} \right) G(k, t) = \delta(k) \delta(t) \quad (22)$$

Using Fourier Transformation:

$$G(k, t) = \int \frac{d\omega}{2\pi} \int \frac{dx}{2\pi} e^{i(kx - \omega t)} \tilde{G}(\omega, x) \quad (23)$$

$$\left[ \omega - \frac{1}{2m}(x^2) - ivx \right] \tilde{G}(\omega, x) = 1 \quad (24)$$

$$\tilde{G}(x, \omega) = \frac{1}{\omega - \frac{x^2}{2m} - ivx} \quad (25)$$

$$G(k, t) = \int \frac{d\omega}{2\pi} \int \frac{dx}{2\pi} \frac{e^{i(kx - \omega t)}}{\omega - \frac{x^2}{2m} - ivx} \quad (26)$$

The pole  $\omega = x^2/2 + ixv$  is located at upper plane if  $x > 0$ , and located on the lower plane if  $x < 0$ . For  $t > 0$ , we should choose contour integration that integrate along the lower plane, after applying Residue Theorem, we get:

$$G(k, t) = -i \int_{-\infty}^0 \frac{dx}{2\pi} e^{ikx - it[x^2/(2m) + ixv]} \quad (27)$$

$$= -ie^{-\frac{im}{2t}(vt+ik)^2} \int_{-\infty}^0 \frac{dx}{2\pi} e^{\frac{-it}{2m}[x - \frac{m}{it}(tv+ik)]^2} \quad (28)$$

Let  $y = x - (-imv + mk/t)$

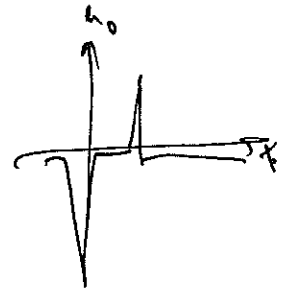
$$G(k, t) = -ie^{-\frac{im}{2t}(vt+ik)^2} \int_{-\infty}^{imv - mk/t} \frac{dy}{2\pi} e^{-ity^2/(2m)} \quad (29)$$

For  $0 < t \ll 1$

$$G(k, t) = (-i) \sqrt{\frac{m}{2\pi it}} \theta(-k) e^{-im(tv+ik)^2/(2t)} \quad (30)$$

$$u_t - 6u u_x + u_{xxx} = 0$$

$$u(x,t)|_{t=0} = -V_1 \delta(x) + V_2 \delta(x-x_0)$$



First, find the eigenvector and eigenenergy of the Schrödinger Equation:

$$\boxed{-\frac{\partial^2}{\partial x^2} \psi + u(x,0) \psi = E \psi}$$

$$\psi(x) = \begin{cases} e^{x\lambda} & x < 0 \\ Ae^{x\lambda} + Be^{-x\lambda} & 0 < x < x_0 \\ Ce^{-x\lambda} & x > x_0 \end{cases}$$

continuity  $A+B=1$  ①

$$Ae^{x_0\lambda} + Be^{-x_0\lambda} = Ce^{-x_0\lambda}$$
 ②

integrating over  $\epsilon < x < \epsilon$   $\epsilon \rightarrow 0$ :

$$\frac{\partial \psi}{\partial x} \Big|_{x=\epsilon} - \frac{\partial \psi}{\partial x} \Big|_{x=-\epsilon} = +V_1$$

↓

$$\lambda - A\lambda + B\lambda = V_1$$
 ③

and at:  $x = x_0 \pm \epsilon$ :

$$\frac{\partial \psi}{\partial x} \Big|_{x=x_0-\epsilon} - \frac{\partial \psi}{\partial x} \Big|_{x=x_0+\epsilon} = -V_2 \cdot Ce^{-\lambda x_0}$$



$$A(A e^{x_0 \lambda} + B e^{-x_0 \lambda}) + \lambda C e^{-x_0 \lambda} = -V_2 C e^{-x_0 \lambda}$$

$$\lambda(A e^{2x_0 \lambda} - B + C) = -V_2 \cdot C \quad (4)$$

We anticipate that  $\lambda \approx V_1$  and:

$$x_0 \lambda < x_0 V_1 \ll 1$$

So we can expand the exponents:

$$A + B + x_0 \lambda(A - B) \quad (2) \qquad A + B = 1 \quad (1)$$

$$= C(1 - x_0 \lambda)$$

$$\lambda(A - B + C + \cancel{2x_0 \lambda A}) \quad (4) \qquad \lambda(1 + B - A) = V_1 \quad (3)$$

$$2x_0 \lambda A = -V_2 C$$

combining (1) and (2):

$$1 + x_0 \lambda(-2B + 1) = C(1 - x_0 \lambda)$$

(1) and (3):

$$\lambda \cdot 2B = V_1$$



$$1 + x_0 \lambda \left(1 - \frac{V_1}{\lambda}\right) = C(1 - x_0 \lambda) \quad (5)$$

and (4):

$$\lambda \left( \underbrace{1 - \frac{V_1}{2\lambda}}_{A-B} + C + 2X_0\lambda \left( 1 - \frac{V_1}{2\lambda} \right) \right) = -V_2 C$$

$$\lambda - V_1 + X_0\lambda(2\lambda - V_1) = -C(\lambda + V_2) \quad (6)$$

neglecting  $X_0\lambda$  we have:

$$C = \frac{V_1 - \lambda}{\lambda + V_2}$$

but keeping every thing,  $\frac{(6)}{(5)}$  gives:

$$\frac{1 + X_0(\lambda - V_1)}{(\lambda - V_1)(1 + X_0\lambda) + \cancel{2\lambda^2}X_0} = -\frac{(1 - X_0\lambda)}{\lambda + V_2}$$

If  $X_0 = 0$ :

$$\lambda + V_2 = -1 \cdot (\lambda - V_1)$$

$\downarrow$

$$\lambda_0 = \frac{V_1 - V_2}{2}$$

let's keep first order:

$$\boxed{\lambda = \lambda_0 + X_0 \delta \lambda}$$

$$\frac{1 + X_0(\lambda_0 - V_1)}{(\lambda_0 - V_1 + X_0 \delta \lambda)(1 + X_0 \lambda) + \lambda_0^2 X_0} = - \frac{1 - X_0 \lambda_0}{\lambda_0 + V_2 + X_0 \delta \lambda}$$

↓

$$\frac{1 + X_0 \left( \frac{-V_1 - V_2}{2} \right)}{- \left( \frac{V_1 + V_2}{2} \right) (1 + \lambda_0 X_0) + \lambda_0^2 X_0 + X_0 \delta \lambda} = - \frac{1 - X_0 \lambda_0}{\frac{V_1 + V_2}{2} + \delta \lambda X_0}$$

take one-over:

$$\frac{X_0 \delta \lambda}{1 - \frac{V_1 + V_2}{2} X_0} + \frac{X_0 \delta \lambda}{1 - X_0 \lambda_0} = - \left( \frac{V_1 + V_2}{2} \right) \cdot \frac{1}{(1 - X_0 \lambda_0)} + \frac{\left( \frac{V_1 + V_2}{2} \right) (1 + \lambda_0 X_0) - \lambda_0^2 X_0}{1 - \frac{V_1 + V_2}{2} X_0}$$

keeping only first order: (in  $X_0$ )

$$2 X_0 \delta \lambda = - \left( \frac{V_1 + V_2}{2} \right) \cdot X_0 \left( \frac{V_1 - V_2}{2} \right) + \left( \frac{V_1 + V_2}{2} \right)^2 X_0 + \left( \frac{V_1 + V_2}{2} \right) \left( \frac{V_1 - V_2}{2} \right) X_0 - \frac{(V_1 - V_2)^2}{4} X_0$$

$$2X_0\delta\lambda = \frac{X_0}{4} \left[ V_2^2 - V_1^2 + V_1^2 - V_2^2 + (V_1 + V_2)^2 - (V_1 - V_2)^2 \right]$$

$$= \frac{X_0}{4} \cdot 4V_1V_2$$

$$\boxed{\delta\lambda = \frac{V_1V_2}{2}}$$

So,  $E = \lambda^2 = (\lambda_0 + X_0\delta\lambda)^2 \approx \lambda_0^2 + 2\lambda_0\delta\lambda X_0$

$$= \frac{(V_1 - V_2)^2}{2^2} + (V_1 - V_2) \cdot \frac{V_1V_2}{2} \cdot X_0$$

So, the instantaneous this predicts ~~are~~ is:

velocity:

$$u(x,t) = - \frac{v/2}{\cosh^2 \left[ \frac{\sqrt{v}}{2} (x - vt) \right]}$$

$$= - \frac{2\lambda}{\cosh^2 \sqrt{\lambda} (x - 4\lambda)}$$

$$v = 4\lambda$$

$$= \frac{(V_1 - V_2)^2}{2}$$

$$+ 2V_1V_2X_0(V_1 - V_2)$$