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A. Inverse scattering method II: Evolution far from the action

What do we do with this? It seems that instead of solving an initial-value problem in u directly, we can first solve the Schrödinger equation for ψ with u as a potential, evolve $\psi(x,t)$ in time, which is a completely linear equation, given that we know the Hamiltonian $\mathcal{H}$ (see last set of notes). Then, if we are really smart, we can figure out what the scattering potential at time t is given that we know $\psi(x,t)$. The latter part is called inverse scattering.

Before me stating the inverse scattering formalism, let's do a bit 'quick and dirty' inverse scattering. If we have a potential $u(x)$ (which we think about as an initial condition for $u(x,t)$), the Schrödinger operator $-\frac{\partial^2}{\partial x^2}+u(x)$, with eigenvalue spectrum λ, will have a spectrum which separates into bound states:

$$\lambda_n < 0,\ \psi_n \sim e^{-\sqrt{|\lambda_n|}|x|},\ |x| \to \infty \tag{1}$$

and propagating states:

$$\lambda_k > 0,\ \psi_k = \begin{cases} a_k e^{-ikx} & x \to -\infty \\ e^{-ikx} + b_k e^{ikx} & x \to \infty \end{cases} \tag{2}$$

and its complex conjugate.

Let us now evolve each one of these solutions according to our Hamiltonian:

$$\psi_t = -i\mathcal{H}\psi \tag{3}$$

But before writing everything out, we should realize that we are interested int he evolution of the solution far from the place where u is finite. In fact, the only thing we know is what is going on far from the potential. That actually makes things simpler. The Hamiltonian at $x \to \infty$ is just:

$$i\mathcal{H} \approx 4\frac{\partial^3}{\partial x^3} \tag{4}$$

Therefore:

$$\psi_t = -4\frac{\partial^3 \psi}{\partial x^3} \tag{5}$$

and, plugging in:

$$\psi = \begin{cases} a_k e^{-4ik^3 t} e^{-ikx} & x \to -\infty \\ e^{-ikx}e^{-4ik^3 t} + b_k e^{ikx} e^{4ik^3 t} & x \to \infty \end{cases} \tag{6}$$

In order to interpret this as a scattering problem, let's keep the oncoming wave constant, and multiply the entire thing by $e^{4ik^3 t}$. This gives:

$$b_k(t)e^{ikx} = b_k e^{ik(x+8k^2 t} \tag{7}$$

This is exactly as if the scatterer is moving to the left with velocity $v = -4k^2$. The interpretation of this is that if some potential landscape u is back scattering at wave-number k, it must be moving to the right at velocity $-4k^2$.

Even more important is the effect that time evolution has on the bound states. In this case, we note that by keeping $k = \sqrt{\lambda}$, with $\lambda < 0$, we have:

$$ikx = i^2\sqrt{|\lambda|}x = -\sqrt{|\lambda|}x.$$

Therefore, the bound state can be thought of as consisting of the reflected wave on the right, and the transmitted wave on the left:

$$\lambda_n < 0,\ \psi_n \sim \begin{cases} b_n e^{-\sqrt{|\lambda_n|}x} & x \to \infty \\ a_n e^{\sqrt{|\lambda_n|}x} & x \to -\infty \end{cases} \tag{8}$$

under this notation we have:

$$b_n(t)e^{-\sqrt{|\lambda|}x} = b_n e^{-\sqrt{|\lambda|}(x-4|\lambda|t)} \tag{9}$$

and:

$$a_n(t)e^{\sqrt{|\lambda|}x} = a_n e^{\sqrt{|\lambda|}(x-4|\lambda|t)} \tag{10}$$

already we see this looks as if the confining potential that is holding the bound state is moving to the *right* with velocity

$$v = 4|\lambda|. \tag{11}$$

In order to conform with the solution for the propagating waves, let's rescale the solution such that a_n is constant, by multiplying by $e^{4|\lambda|^{3/2}t}$:

$$a_n(t) = a_n, \;\; b_n(t) = b_n e^{8|\lambda|^{3/2}t}. \tag{12}$$

The general structure of the solution is becoming apparent. Every bound state of the initial potential $u(x)$, will be bound to an object propagating to the right. Furthermore, each such object will have a different velocity: $v_n = 4|\lambda_n|$. Each one of these is a soliton like the one we found as the traveling solution. In fact, we know exactly the form:

$$u_n(x,t) = -\frac{2|\lambda_n|}{\cosh^2\left(\sqrt{|\lambda_n|}(x-4|\lambda_n|t)\right)}. \tag{13}$$

The deeper the bound state energy is, the deeper and faster the soliton is.

What we achieved is almost what we sought out: we now know essentially that a soliton with velocity v is associated with a bound state of velocity $4|\lambda_n|$. So to understand how many solitons emanate from the initial data, all we need to do is figure out how many bound states there are. There are also other things that emanate from the initial data, which move to the left rather than to the right (which the solitons do). These are somehow associated with the positive energy unbound solutions of the Schrödinger equation. Next, we get a recipe that will tell us what the scattering potential is.

B. Inverse scattering method III: The Marchenko equation

The problem that we would like to solve is: Given a scattering data $b_k(t)$ for the traveling waves, and $b_n(t)$, λ_n for the bound states, what is the scattering potential $u(x,t)$?

The answer is given by three not too complicated to state equations. First, define a function $B(x,t)$ which is an accumulation of all the things that we know:

$$B(x+y,t) = \sum_n b_n(t)e^{-\sqrt{|\lambda+n|}(x+y)} + \int \frac{dk}{2\pi|} b_k(t)e^{ik(x+y)} \tag{14}$$

Define another operator $K(x,y,t)$ which is zero for $y < x$. Now, here's the kicker. If $K(x,y,t)$ is the solution of the integral equation:

$$K(x,y,t) + B(x+y,t) + \int_x^\infty dz K(x,z,t)B(z+y,t) = 0 \tag{15}$$

then the scattering potential is:

$$u(x,t) = -2\frac{\partial K(x,x,t)}{\partial x}. \tag{16}$$

The above integral equation is called the Marchenko equation.