September 30, 2010

ph129, 9/29/10 - Scaling method for elimination of a variable in a PDE

I. GENERAL ADMINISTRATIVE COMMENTS

1. Website:

   For the first portion of the class:

   www.cmp.caltech.edu/˜refael/ph129/

   In general, Frank Porter's page is also available, and provides helpful material for all topics in the class.

   www.hep.caltech.edu/˜fcp/ph129/index.shtml

2. Book. There is no specific course book we will follow. A great resource for the first portion is Mathematics for Physics by Stone and Goldbart.

3. Grading. The course grad will consist of 40% homework, 25% midterm grades, and 35% final grades.

4. Plan. For the first half we will concentrate on differential equations in all possible guises. We will start with simple PDE's, withdraw a bit to complicated ODE's, and then go non-linear.

Simple PdE's: time and single space dimension

II. DIFFUSION EQUATION

The diffusion equation is given by:

$$\frac{\partial \rho}{\partial t} = D\nabla^2\rho. \tag{1}$$

We will first discuss its derivation. Then general methods of attack. We will expand on one of them (scaling), and use it to derive a special solution. We'll then use the solution to solve the inhomogeneous diffusion equation, and the homogeneous boundary-value problem. We'll close by using a more conventional approach to the problem.

A. Derivation

The first question we must ask is where does this equation come from? Knowing the physics behind equations is often extremely helpful in understanding how to solve them.

Let us give a mock derivation of this equation using some drawings on the board and some hand-waving motion. Consider a gas of diffusing particles. By this I mean particles that move through a disordered medium and keep colliding with impurities and with themselves. We are interested in the evolution of their density on time scales and length scales longer than the mean free path and time.

Consider some density profile. Where would current flow? Down from high density to low density.

$$\vec{j} = -D\nabla\rho. \tag{2}$$

How do we connect the current, which we now know in terms of the density gradient, back to the density? The answer lies in the mass (or particle) conservation low:

$$\nabla \cdot \vec{j} = -\frac{\partial \rho}{\partial t}. \tag{3}$$

If you don't know where this came from think about how the density changes in a small box with currents coming in and out of it, and you'll immediately see.

Putting the two equations together we get:

$$-\frac{\partial \rho}{\partial t} = \nabla \cdot (-D\nabla\rho) = -D\nabla^2\rho. \tag{4}$$

B. Finding solutions

Differential equations are never as innocent as they look. And it is only in few cases that we can analytically get a handle on their solution. Let's think for a second, how should we go about finding a solution of, say, the diffusion equation, or any other *partial* differential equation?

1. Guess. Sometimes will have to simply try to come up with our best guess. No shame in that. In this class we'll try to give you some tools that will reduce the amount of guessing necessary.

2. Separation of variables. Indeed, this is the most common way of dealing with PDE's. The idea is the following guess (illustrated in 2d):

$$\rho(x,t) = R(x)\theta(t). \tag{5}$$

Substituting in the diffusion equation, for instance, we get:

$$R(x)\frac{\partial\theta(t)}{\partial t} = D\theta(t)\frac{\partial^2 R(x)}{\partial x^2}, \tag{6}$$

which immediately should be written as:

$$\frac{1}{\theta(t)}\frac{\partial\theta(t)}{\partial t} = D\frac{1}{R(x)}\frac{\partial^2 R(x)}{\partial x^2} = \text{const.} \tag{7}$$

The last observation is the important one: since both sides of the equation depend on different independent variables (x and t), they must equal a constant. We use this a lot - you must have seen it before somewhere. We will use this heavily when we use green functions.

3. Scaling. As it turns out, there is a surefire way of reducing the number of independent variables by one. It involves transforming all variables into scale invariant variables. We will explore this approach now.

C. Scaling variables

It is almost ironic that the biggest simplification the scaling method affords is achieved without use of calculus or algebra for that matter. It is almost purely conceptual. Here's the idea: we have a problem given in terms of ρ which is a function of x and t. But now, instead, we would like to look at this equation, in terms of a different grid. Say we carry out the following *scaling transformation*:

$$t \to t' = t/b^z,\ x \to x' = x/b \tag{8}$$

So if $b > 1$, x' and t' are a more coarse measure - less refined. It is not clear what this is good for yet, and further, we have this free variable - z.

To determine z we require that: *the PDE at hand in invariant under the scaling transformation.* We notice:

$$\frac{\partial}{\partial t} = \frac{1}{b^z}\frac{\partial}{\partial t'} \quad \frac{\partial}{\partial x} = \frac{1}{b}\frac{\partial}{\partial x'} \tag{9}$$

so the diffusion equation becomes:

$$\frac{1}{b^z}\frac{\partial\rho}{\partial t'} = \frac{1}{b^2}D\frac{\partial^2\rho}{\partial x'^2}. \tag{10}$$

Choosing

$$z = 2$$

fits the bill.

This, however, is not enough. We have another variable that we did not touch - ρ. We also can have:

$$\rho \to \rho' = \rho/b^\psi. \tag{11}$$

How should we determine ψ? In the PDE it cancels.

The trick here is that in addition to the differential equation, ρ has a very specific meaning - density - and it obeys a conservation law. We can see this by integrating over space from $x = -\infty \to \infty$:

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} = D\left.\frac{\partial \rho}{\partial x}\right|_{-\infty}^{\infty} = 0 \tag{12}$$

which means:

$$\int_{-\infty}^{\infty} \rho = constant \tag{13}$$

It makes sense to require that the constant is also scale invariant. This gives the requirement:

$$\rho dx \to \rho' \Delta'_x = b^{-\psi-1}\rho dx = \rho dx \tag{14}$$

so

$$\psi = -1 \tag{15}$$

makes sense: $x \to x' = x/b$, and therefore $\rho \to \rho' = b\rho$, so the total mass is conserved in the transformation.

Now, the scaling trick is as follows: *We can eliminate one variable from the PDE by using scale invariant variables.* Of course we cant have as many new scale-invariant variables as we had regular variables. We must maintain one variable as the original one. Choose:

$$\bar{t} = t \tag{16}$$

What are the remaining scale invariant variables? [Ask students!]

$$\zeta = x/t^{1/z} = x/\sqrt{t} \tag{17}$$

By subbing x' and t' we see this is indeed scale invariant. And:

$$\phi = \rho/t^{\psi} = \rho t. \tag{18}$$

Now, the trick is that we expect that ϕ is only a function of ζ, $\phi(\zeta)$.

The next step is to transform our differential equation to these new variables. For that we need some prep work. Let's deal with x first. Clearly, when x changes with t staying constant, only $\zeta = x/\sqrt{t}$ changes. So clearly:

$$\frac{\partial}{\partial x} = \frac{\partial \zeta}{\partial x}\frac{\partial}{\partial \zeta} = \frac{1}{\sqrt{\bar{t}}}\frac{\partial}{\partial \zeta} \tag{19}$$

where we already used the overline t.

When we change t both zeta and $\bar{t}$ change. Therefore we must use the complex rules for changing variables in a partial derivative:

$$\left.\frac{\partial}{\partial t}\right|_x = \left(\frac{\partial \bar{t}}{\partial t}\right)_x\left(\frac{\partial}{\partial \bar{t}}\right)_\zeta + \left(\frac{\partial \zeta}{\partial t}\right)_x\left(\frac{\partial}{\partial \zeta}\right)_t = \left(\frac{\partial}{\partial \bar{t}}\right)_\zeta - \frac{\zeta}{2t}\left(\frac{\partial}{\partial \zeta}\right)_t. \tag{20}$$

Putting this together we get:

$$\left(\frac{\partial \phi(\zeta)/\sqrt{t}}{\partial \bar{t}}\right)_\zeta - \frac{\zeta}{2t}\left(\frac{\partial \phi(\zeta)/\sqrt{t}}{\partial \zeta}\right)_t \tag{21}$$

which becomes:

$$\frac{\partial^2 \phi}{\partial \zeta^2} + \frac{1}{2}\zeta\frac{\partial \phi}{\partial \zeta} + \frac{1}{2}\phi = 0 \tag{22}$$

No $\bar{t}$!