Problem set - 4

Due date: Monday, Dec 3^{rd}, 8pm.

1. Pauli Susceptibility. In this problem you will calculate the Pauli susceptibility, which is the susceptibility of a free spinfull fermionic gas, $\partial M/\partial H$, where M is the magnetization of the gas and H is the external magnetic field.

   (a) In the presence of a magnetic field H, the energy of a state changes depending on its spin. In addition to the kinetic energy there will be a magnetic energy :

   $$\mathcal{H}_{spin} = -\frac{1}{2} g \mu_B H \sigma_z \tag{1}$$

   that is, for an up/down spin, the energy of a state at momentum $\vec{k}$ becomes $E_k^{\uparrow/\downarrow} = \epsilon_k \mp \frac{1}{2} g \mu_B H$, where ϵ_k is the kinetic energy.
   For a given chemical potential μ, what is the highest kinetic energy of a filled state of spin up and spin down electrons at zero temperature (assume $T = 0$ for the entir problem)?

   (b) What is the magnetization ($M = \frac{1}{2} g \mu_B (n_\uparrow - n_\downarrow)$) of the gas for a given H? assume the magnetic field H is small, and that the density of states $\rho(\epsilon)$ is that of a two dimensional free electron gas ($\epsilon_k = |k|^2/2m$). What is the susceptibility?

2. Electronic heat capacity of a superconductor. The low-T energy spectrum of a metal which turns superconducting is unique. A gap opens up about the fermi surface, and the excitation energies allowed (i.e., the possible energy states, that replace the free-electron energies ϵ_k) are:

   $$E = E_F \pm \sqrt{\Delta^2 + (\epsilon_{\vec{k}} - E_F)^2} \tag{2}$$

   The $\pm$ is determined by the sign of $(\epsilon_{\vec{k}} - E_F)$, such that when $|\epsilon_{\vec{k}} - E_F| \gg \Delta$ the electronic states are recovered. Each state is doubly degenerate due to spin. $\epsilon_{\vec{k}}$ should be thought of as the kinetic energy of the electronic state $\vec{k}$ in the abscence of superconductivity.

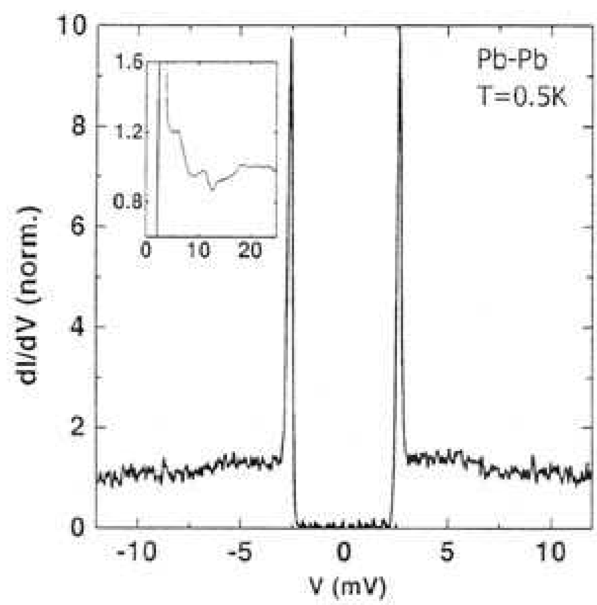

FIG. 1: Tunneling DOS of lead. Taken from H. Suderow, et al., Physica C, vol. 369, pp. 106 (2002).

   (a) Approximate the metallic density of states (in the abscence of superconductivity), $\rho(\epsilon_{\vec{k}}) = \rho_0$, as constant. Show that the DOS of the superconductor is given by:

   $$\rho(E) = \begin{cases} 0 & |E - E_F| < \Delta \\ \rho_0 \frac{|E - E_F|}{\sqrt{(E - E_F)^2 - \Delta^2}} & |E - E_F| \geq \Delta \end{cases} . \tag{3}$$

(b) (ungraded) The density of states is directly measurable using tunneling experiments. In these experiments the differenetial conductance, $\frac{dI}{dV}$ is measured as a function of tunnleing voltage V. The conductivity is linearly proportinal to the DOS at energy eV. Plot the density of states you obtained above, and compare it to Fig. 1.

(c) Use the FD distribution to find an expression for the energy stored in the electronic states assuming $T \ll \Delta$. You can leave the answer in integral form. Assume that as long as the energy is not $|\epsilon_{\vec{k}} - E_F| \ll, \gg \Delta$, the density of states remains constant. For the rest of the range express your result in terms of a function $\rho(\epsilon_{\vec{k}})$ and leave it in integral form.

(d) What is the electronic heat capacity of the superconductor? Find the leading behavior of the heat capacity as a function of temperature upto numerical factors. Hint: check that only the regions of the spectrum closest to the fermi-energy contribute.

3. Entanglement entropy of a random spin chain. A random Heisenberg model dscribes a chain of spin-1/2 sites, interacting with their nearest neighbor in an isotropic antiferromagnetic way, but with an energy scale that differs from bond to bond:

$$\hat{\mathcal{H}} = \sum_i J_i \hat{\vec{S}}_i \hat{\vec{S}}_{i+1} \tag{4}$$

It was found that the ground state of this model consists of singlets forming an a random fashion between pairs of spin, as shown in Fig. 2. Crudely speaking, these singlets form over strong bonds, but can also be long range - this is a remarkable result of quantum fluctuations. A topic of recent investigation is the entanglement entropy between two parts of the chain.

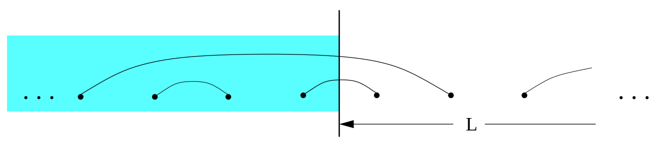

FIG. 2: In the random singlet ground state, pairs of sites form singlets in a random fashion. These singlets mostly connect nearest neighbors, but also connects spins at arbitrarily large distances. On average, the correlations in this phase decay as $1/r^2$ (with respect to distance$r$)- very slowly. In this problem, we are interested in the entanglement between two parts of the chain - the shaded part, and the unshaded part. For that purpose we calculate the reduced density matrix for the non-shaded part, by tracing over the shaded one.

The *entanglement entropy* of two subsystems, A and B, is defined as:

$$S = -tr\hat{\rho}_A \log_2 \hat{\rho}_A \tag{5}$$

(defines with the information-theory afficionados log base-2) here $\hat{\rho}_A$ is the reduced density matrix, $\hat{\rho}_A = tr_B\hat{\rho}$, and the density matrix is the operator:

$$|\psi\rangle \langle\psi| . \tag{6}$$

Note that it doesn't matter whom we call A and whom B:

$$S = -tr\hat{\rho}_A \log_2 \hat{\rho}_A = -tr\hat{\rho}_B \log_2 \hat{\rho}_B. \tag{7}$$

(a) For illustrative purposes, assume here that after tracing $\hat{\rho}$ over subsystem B, the resulting reduced density matrix $\hat{\rho}_A$ describes a mixed state:

$$\hat{\rho}_A \to \sum_i p_i |i\rangle \langle i| \tag{8}$$

with $\sum\limits_i p_i = 1$ and $\langle i| j\rangle = \delta_{ij}$. What is the entanglement entropy between A and B?

(b) Suppose that once B is traced, the reduced density matrix can be written as:

$$\hat{\rho}_A = \hat{\rho}_{A_1} \otimes \hat{\rho}_{A_2} \otimes \ldots \otimes \hat{\rho}_{A_n} \tag{9}$$

Where the A_i's describe the degrees of freedom, or Hilbert space, of mutually exclusive subdivisions of A (in other words, the Hilbert space of A is a product space of the Hilbert spaces of the A_i's). You can think of the index i as a site index - the density matrix factorizes into a product of the density matrices for different sites. Show that the entanglement entropy is now:

$$S = -\sum_{i=1}^{n} tr\hat{\rho}_{A_i} \log_2 \hat{\rho}_{A_i} \tag{10}$$

(use the fact that *log* of operator products behave as *logs* of number products).

(c) Concentrate on a single singlet between sites 1 and 2:

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2 \right). \tag{11}$$

First, show that the entanglement calculated for the density matrix $\hat{\rho}_A = |\psi\rangle \langle\psi|$ (this is the reduced density matrix in case sites 1 and 2 are contained in part A) is zero.

Next, show that the entanglment entropy between spin 1 and spin 2 is 1 (think of site 1 as part of A and site 2 as part of B), Since each spin represents one qubit, this implies that they are completely entangled.

(d) Show explicitly that the entanglement between two sides of the chain described by the random-singlet ground state is given by the number of singlets connecting the two sides.

To read more about this problem see https://arxiv.org/abs/cond-mat/0406737 and references therein.

4. Calculting the Sommerfeld expansion for a 1-d band. The dispersion of electrons in real metals often involves a very complicated band structure. Consider the following one-dimensional single band toy model. The dispersion of electrons confined to a 1-d wire is given by:

$$\epsilon_{\vec{k}} = -t \cos ka \tag{12}$$

where a is length-scale signifying a lattice constant, and t has the meaning of a tight-binding hopping between neighboring lattice sites, and $-\pi \leq ka < \pi$. Note that the kinetic energy is $-t$ at the bottom of the band; if you wish, you can shift this to be zero - but if you do, state it clearly in your solution, otherwise absolute energies would be assumed.

(a) If the band is half full, i.e., half of the quantum states of the band are filled and half are empty, what is the spatial electronic density? Assume that state counting in k-space is as was discussed in class.

(b) what is the Fermi energy of the system? (Note that the chemical potential at low-T, and the Fermi-energy, are only restricted to be higher than the bottom of the band)

(c) What is the density of states $\rho(\epsilon)$? Hint: if you are not comfortable with δ-functions, use the differential definitial. Sketch your the result in a graph.

(d) Find the total energy of the electrons in the band upto fourth order in temperature, T, assuming μ is held constant at the Fermi energy.

(e) Find the heat capacity of the gas to third order in T assuming $\mu = E_F$ and is held constant.

(f) If we hold the chemical potential fixed at the Fermi-energy E_F, what is the number of particles as a function of temperature? Find an answer valid to all orders in T.

5. White dwarves. The theory of degenerate Fermi-gases provides a good description of stars that are abnormally faint although their apparent color is white - white dwarves. Their low brightness could be explained by the stars using up all the hydrogen supply, and we can consider them as consisting of just Helium. The radius of the star, it was contended by Chanderasekhar, is obtained through equilibrium between the degeneracy pressure of the electrons, and the gravitational pull of the Helium.

Consider a star that has a mass M and consists purely of helium atoms (with atomic mass $6.4 \cdot 10^{-27}$). The temperature of the star is typically $10^7 K$, but nevertheless we assume that this temperature is much smaller than the Fermi energy of the electrons in the star.

(a) If the radius of the star is R, what is the fermi energy of the electrons in it? Assume the electrons are non-relativistic, and are not bound to the Helium nuclei . Express your answer in terms of M and R, and whatever other physical constants you deem necessary.

(b) What is the total energy of the electron gas?

(c) What is the gravitational energy of the star? No need to calculate it exactly - express it upto numerical factors of order one in terms of M and R. You may find the gravitaional constant $G = 6.66 \cdot 10^{-11} m/Jkg^2$ useful. Make sure to get the correct sign - the star wants to graviationally collapse.

(d) Minimize the total energy (gravitational + electronic) to find the equilibrium volume of the star. What is this radius? What is the Fermi energy of the electrons? How does it compare to the rest mass energy of the electrons ($m_e c^2$)? Typically, the mass of white dwarves is of order of the sun's mass: $10^{30} kg$, use this to obtain numerical values for this part only.

(e) From 5d you should infer that the electrons are actually *not* adequately described in terms of classical mechanics. Assume the opposite limit, of ultra-relativistic electrons, with $E = pc = c\hbar k$ as their kinetic energy. What is the energy of the gas assuming the electrons are ultra-relativistic? Show that this energy has the same dependence on the radius of the star as the gravitational energy.

(f) From the conditions of stability infer a condition on the mass of a star such that it would form a white dwarf, rather than explode into a supernova, or a neutron star. The critical mass you find here is the Chanderasekhar mass. The refined result is $M_C = 1.44 M_{sun}$.