Ph127a
Solutions to Problem Set - 5

1. (a) The number of particles as a function of $\zeta = e^{\mu\beta}$ is

$$\frac{N}{V} = \int \frac{d^d k}{(2\pi)^d} \sum_{n=1}^{\infty} e^{-n|k^2|^2/2\alpha T} \zeta^n \tag{1}$$

This came from expanding the BE distribution in powers of ζ. Rescaling k leads to:

$$\begin{aligned} \frac{N}{V} &= \left(\int \frac{d^d x}{(2\pi)^d} e^{-x^4/2} \right) (\alpha T)^{d/4} \sum_{n=1}^{\infty} \frac{\zeta^n}{n^{d/4}} \\ &= C_d \, (\alpha T)^{d/4} g_{d/4}(\zeta) \end{aligned} \tag{2}$$

the function $g_q(\zeta)$ is the sum, C_d is the integral term, which is just a number (it does not diverge nor vanish for all d). As we lower the temperature, the $(\alpha T)^{d/4}$ term approches zero, so for the expression to remain adequate for describing a finite density, $g_{d/4}(\zeta)$ must diverge for $\zeta = 1$. This happens when $d/4 > 1$, so the lowest dimension for BEC is $d = 5$.

(b) The critical temperature is defined by the above expression with $g_{5/4}(\zeta)$ at its maximal value $g_{5/4}(1) \approx 4.6$:

$$T_{BEC} = \frac{1}{\alpha} \left(\frac{n}{4.6 C_5} \right)^{4/5} \tag{3}$$

(c) This spectrum shifts the minimun energy states from $\vec{k} = 0$ to $|\vec{k}| = k_0$. The energy spectrum near the minimum is approximately parametrized by a single parameter k_1, and is:

$$\epsilon_k \propto k_1^2 \tag{4}$$

This is the same as the one-dimensional dispersion, and therefore in this problem, any dimension looks as though it is one-dimensional with respect to the low energy spectrum. So in all dimensions there is a finite density of states at the minimum energy, and therefore BEC does not exist for any dimension.

2. (a) The grand canonical partition function is generally:

$$\mathcal{Z} = \sum_{N=0}^{\infty} Z_N \zeta^N \tag{5}$$

where Z_N is the partition function of N particles (including the integrals over real space), and $\zeta = e^{\mu\beta}$. We can expand this expression to second order in ζ:

$$\mathcal{Z} \approx 1 + \zeta Z_1 + \zeta^2 Z_2 + \mathcal{O}(\zeta^3) \tag{6}$$

When the system has only one particle, there are no interactions, and we have:

$$Z_1 = \int \frac{d^3x d^3p}{h^3} e^{-p^2/2mT} = V \left(\frac{\sqrt{2\pi m T}}{h} \right)^3 \tag{7}$$

To make the dependence on volume apparent, let's redefine Z_1 to be the partition function without the V :

$$Z_1 = \left(\frac{\sqrt{2\pi m T}}{h} \right)^3 = \frac{1}{\lambda_T^3} \tag{8}$$

For the two particle partition function, the momentum integrals are the same as the one particle case:

$$Z_2 = \frac{1}{2} Z_1^2 \int d^3 r_1 \int d^3 r_2 e^{-V(r_{12})/T} \tag{9}$$

The factor of 1/2 is because the particles are identical. To this order we have:

$$\mathcal{Z} \approx 1 + \zeta Z_1 V + \zeta^2 \frac{1}{2} Z_1^2 \int d^3 r_1 \int d^3 r_2 e^{-V(r_{12})/T} \tag{10}$$

To second order in ζ this expression is equal to:

$$\mathcal{Z} \approx exp\left(\zeta Z_1 V + \frac{1}{2!}\zeta^2 Z_1^2 \int d^3 r_1 \int d^3 r_2 \left(e^{-V(r_{12})/T} - 1\right)\right) + \mathcal{O}\left(\zeta^3\right), \tag{11}$$

Notice the (-1) in the exponent is needed to get the two expression equal to second order in ζ

(b) We start from the the grand canonical potential:

$$\omega = -T \ln \mathcal{Z} \approx -T\left(\zeta Z_1 V + \frac{1}{2!}\zeta^2 Z_1^2 \int d^3 r_1 \int d^3 r_2 \left(e^{-V(r_{12})/T} - 1\right)\right) \tag{12}$$

We can expose the volume dependence by changing variables to $r_{12} = r_1 - r_2$ and $R = (r_1 + r_2)/2$. The Jacobian of this transformation is unity, and since the interaction does not depend on R, we integrate it out to give a factor of V:

$$\omega = -T \ln \mathcal{Z} \approx -T\left(\zeta Z_1 V + \frac{1}{2} Z_1^2 \zeta^2 V \int d^3 r_{12} \left(e^{-V(r_{12})/T} - 1\right)\right) \tag{13}$$

Let's define :

$$\int d^3 r_{12} \left(e^{-V(r_{12})/T} - 1\right) = C_{12} \tag{14}$$

The grand canonical potential is simply $(-pV)$, so we can read off the pressure:

$$p = T\left(Z_1 \zeta + \frac{C_{12}}{2}(Z_1 \zeta)^2\right) \tag{15}$$

The average number of particles is:

$$N = -\frac{\partial \omega}{\partial \mu} = T\left(Z_1 V \zeta + C_{12} V (Z_1 \zeta)^2\right) \tag{16}$$

we need to invert eq. (16) to get $Z_1\zeta$ as a function of N/V and substite this into eq. (15). To first order $Z_1\zeta = N/V = n$, so we write an expansion compatible with this fact:

$$Z_1 \zeta = n + A n^2 + \mathcal{O}(n^3) \tag{17}$$

putting eq.(17) in eq.(16) and we see that for both side to be equal for every power of ζ, we must have

$$A = -C_{12} \tag{18}$$

Now plugging the expansion for $Z_1\zeta$ into the expression for pressure we get:

$$p = T\frac{N}{V}\left(1 - \frac{C_{12}}{2}\left(\frac{N}{V}\right)\right) \tag{19}$$

This is a good place to evalute C_{12}:

$$\begin{aligned} C_{12} &= \int d^3 r_{12} \left(e^{-V(r_{12})/T} - 1\right) = 4\pi \int_0^\infty r^2 \left(e^{-V(r)/T} - 1\right) dr \\ &= -4\pi \int_0^b r^2 dr + 4\pi \int_b^\infty r^2 \left(e^{-a/Tr^6} - 1\right) dr \end{aligned} \tag{20}$$

$$\tag{21}$$

for high temperature, $T \ll a/b^6$, we can expand the exponential term to first order in $1/T$:

$$C_{12} \;\approx\; -\frac{4\pi}{3}b^3 - 4\pi\int_b^\infty dr\frac{a}{T}r^{-4} = -\frac{4\pi}{3}b^3 + \frac{4\pi a}{3T}b^{-3} \tag{22}$$

Eq.(19) corresponds to the Van der Walls equation of state, which is usually written as:

$$\left(p + \alpha\frac{N^2}{V^2}\right)(V - \beta N) = NT \tag{23}$$

but where $1/(V - B\beta)$ is expanded for small $N\beta/V$. In our case $\alpha = 4\pi a/b^3$ and $\beta = 4\pi b^3/3$.