

Due date: Feb 19th.

Ph 127b - Problem set 2

Note - if you're looking to break this down into two doable one week problem sets, do 1-3 this weekend, and 4-5 next one.

1. Weiss theory of an antiferromagnet.

Consider a square 2-d lattice with an Ising spin, $\sigma_i = \pm 1$, on each site, and with a nearest-neighbor Hamiltonian:

$$\hat{\mathcal{H}} = J \sum_{\langle ij \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i \quad (1)$$

Note that the interaction encourages nearest-neighbors to point in opposite directions.

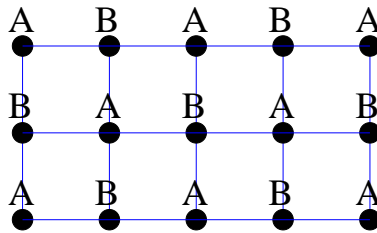


FIG. 1: A square lattice can be broken down to two sublattices, A and B with interaction only occurring between sublattices.

The square lattice is a *bipartite lattice*. Its sites can be divided into A and B sublattices, such that sites in A only interact with sites in B and vice versa (See Fig. 1). To construct a mean-field solution of this model we assume that the material forms a *staggered* or *Ne'el* magnetization pattern:

$$\langle \sigma_j \rangle = \begin{cases} m & j \in A \\ -m & j \in B \end{cases} \quad (2)$$

- Find a self consistent equation for the magnetization m , and solve it for small m 's, for $h = 0$ and as a function of temperature. Does the system exhibit a phase transition? At what temperature?
- What are the broken symmetries in this transition?
- In the Hamiltonian of this system there is an interaction with a uniform (as opposed to staggered) magnetic field. Construct the self consistent equations for the magnetization in the A and B sublattices, and find the susceptibility of the system to a uniform field. Show it is:

$$\chi = \frac{1}{T + T_C} \quad (3)$$

2. Heisenberg ferromagnet in mean-field.

Consider a magnetic material that consists of a cubic lattice in 3-d. In each atom resides a magnetic dipole, which can be considered as classical (rather than quantum). We represent the dipoles as a unit vector, \vec{s}_i for the dipole on site i . If the dipoles interact through a rotationally symmetric, SO(3), ferromagnetic interaction, than the Hamiltonian describing them is:

$$\hat{\mathcal{H}} = -J \sum_{\langle ij \rangle} \vec{s}_i \cdot \vec{s}_j - \vec{h} \cdot \vec{s}_i \quad (4)$$

Where the sum is only on nearest-neighbor bonds.

In this problem we will find the Heisenberg model mean-field phase diagram and properties.

- What is the lower critical dimension of this model? Hint - Mermin-Wagner theorem.

- (b) Assume that the interaction term of spin \vec{s}_i can be replaced with an interaction with an average magnetization $\vec{m} = \langle \vec{s}_j \rangle$. Find a self consistent equation for the magnetization \vec{m} . Assume that $|\vec{m}| \ll 1$ in the region of interest. A rather useless hint: before doing any integration on solid angles, make sure you know the correct Jacobian (or integration measure) for it if you are using polar coordinates.
- (c) Is there a phase transition? At what temperature?
- (d) What is the average magnetization as a function of temperature near the transition?
- (e) What is the Landau free energy that describes this transition (you're not expected to rederive it from scratch - rather use the previous results in this problem to infer it)?
- (f) What is the susceptibility to the field h ?

3. The free energy:

$$F_L(T; m) = a(T - T_C)|\psi|^2 + d|\psi|^6 - \Re(h^* \psi) \quad (5)$$

undergoes a second order phase transition at T_C . Find the following critical exponents of this transition:

- (a) α .
 (b) β .
 (c) γ .
 (d) δ .

4. Domain wall/Liquid-Gas interface and surface tension.

Consider a substance with a Landau free energy given by:

$$F_L(m, T) = \int dx dy dz \left[\gamma (\nabla m)^2 + a(T)m^2 - Cm^3 + bm^4 \right] \quad (6)$$

(note that we are considering the usual 3-d world).

- (a) At a critical value of $a(T)$ the substance undergoes a phase transition between $m = 0$ and $m = m_0$. What is the value of $a(T)$ at this point? Hint - the free energy is the same for $m = 0$ and $m = m_0$. What is m_0 ?
- (b) Show that when $a(T)$ attains its value at the transition, the free energy can be written as:

$$F_L(m, T) = \int d^3x \left[\gamma (\nabla m)^2 + bm^2(m - m_0)^2 \right]. \quad (7)$$

At the transition point the $m = 0$ phase and the $m = m_0$ phases may coexist (consider this as the coexistence of magnetic domains, or as gas-liquid coexistence). In the following we derive properties of the boundary (in space) between the two phases. Assume that the order parameter is only a function of x .

The task of exactly minimizing the free energy at the transition point is tough. Therefore we will use the approximate variational method. We will guess the order-parameter profile, upto one parameter, and then minimize the free energy with respect to that parameter.

Let's assume that the order-parameter exhibits a domain wall at $x = 0$, between $m = 0$ at $x \rightarrow -\infty$ and $m = m_0$ when $x \rightarrow \infty$. A reasonable guess for $m(x)$ is:

$$m(x) = \left(1 + \tanh \left(\frac{x}{\ell} \right) \right) \frac{m_0}{2} \quad (8)$$

ℓ is our variational parameter.

- (c) Show that the free energy as a function of ℓ (before minimization) is:

$$F(\ell) = \frac{1}{3} m_0^2 \left(\frac{\gamma}{\ell} + \frac{m_0^2}{4} \beta \ell \right) \quad (9)$$

- (d) What is the thickness of the domain wall between the two phases? In other words, what choice of ℓ minimizes the free energy?

- (e) By comparing the curvature of the free energy around $m = 0$ (or $m = m_0$) with respect to m , to the gradient term one can obtain a length scale. How does this length scale relate to the domain wall thickness?
- (f) What is the free energy per area (in the y - z plane) of the domain wall? What physical property does this correspond to in the liquid-gas phase transition?

5. Uniform Landau theory for symmetry reduction transition in a square lattice.

A three dimensional solid forms a square lattice with lattice constant a at high temperature. At the temperature T_C a transition occurs, and the lattice becomes tetragonal. This means that one direction in the cube stretches (or shrinks) to size b (see fig. 2).

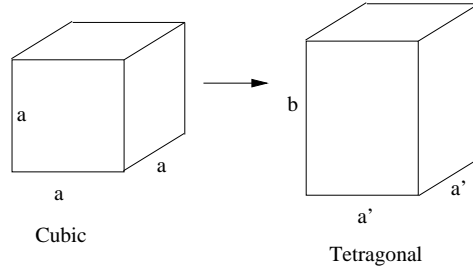


FIG. 2: A cubic lattice undergoes a symmetry breaking transition and becomes tetragonal, with one side longer than the other two.

- (a) What are the point symmetries (i.e. symmetries that leave one point in the lattice in its place) of the original square lattice? Indicate them as simply as you can.

The order parameter for the transition can be chosen to be the deviation of the lattice constant in each direction (x,y,z) from a :

$$\vec{d} = \begin{pmatrix} d_x \\ d_y \\ d_z \end{pmatrix} = \begin{pmatrix} a_x - a \\ a_y - a \\ a_z - a \end{pmatrix} \quad (10)$$

- (b) How do the various parts of the order parameter transform under the symmetry operations of the lattice?
- (c) Show that the following Landau free energy satisfies all the symmetries:

$$F = a(T)(\vec{d})^2 + b(d_x^4 + d_y^4 + d_z^4) - C(d_x^3 + d_y^3 + d_z^3) + s(d_x d_y + d_z d_y + d_x d_z). \quad (11)$$

So far, we usually dropped the cubic term for symmetry considerations. Are we allowed (and therefore required) to include it here? Explain briefly.

- (d) Before approaching the full Landau free energy, find the phase diagram for only the x -direction part of the free energy, ignoring interactions. For simplicity, assume $\frac{da(T)}{dT} > 0$, and is the only coefficient in the free energy which varies with T . At what value of $a(T)$ is there a phase transition? Find d_x directly below and above the transition temperature.
- (e) Let us now analyze the entire field theory. To begin, ignore interactions, what are the points in the 3-d \vec{d} space with lowest free energy at the temperature found in (d)? Now, how does the interaction term between the various directions affect these minimas? What are the surviving minimas?
- (f) From (e), we learn that when one of lattice direction expands, the other two definitely do not. Nevertheless the interaction term induces them to shrink. By assuming that the expanding direction is d_x , and that $d_y = d_z$ following that, find the two self-consistent equations that determine \vec{d} .
Byt using the fact that $d_y = d_z$ are small, neglect terms higher than qadratic in them. By eliminating $d_y = d_z$, find d_x . When is there a phase transition? What are d_x and d_y near the transition above and below it?
- (g) Can you qualitatively explain why the sign of s was chosen as it was? Hint - we're talking about a solid.