Week 1 - Continuous Phase Transitions and Symmetry Breaking

- Phase transition of order n exhibits a singularity or discontinuity only in the n'th or higher derivatives of a thermodynamic potential,

  $$\frac{\partial^n f}{\partial x^n}.$$

  Transition with $n \geq 2$ are referred to as **continuous phase transitions** or **critical points**.

- Examples of second order phase transitions:
  - magnetic systems
  - superconductivity
  - superfluidity
  - structural lattice transitions

- All transitions are between disordered states of high symmetry to ordered states with lower symmetry.

A. The Ising model

A model of spins with interactions. Each spin is $S_i = \pm 1$. The Hamiltonian:

$$\hat{\mathcal{H}} = -\sum_i J\sigma_i\sigma_j \tag{1}$$

- At high temperature phase we have a disordered phase. Low temperature - ordered phase, but with two possible ground states - $|\uparrow\uparrow\uparrow\uparrow\ldots\rangle$ and $|\downarrow\downarrow\downarrow\downarrow\ldots\rangle$. There is a finite average magnetization:

  $$m = \langle\sigma_i\rangle. \tag{2}$$

  and also **Long range order**:

  $$\langle\sigma_i\sigma_j\rangle \sim m^2 \tag{3}$$

  at large distances, $|i-j|$. m is the **Order parameter**.

B. Symmetry and its breaking

Consider a symmetry operation, $\mathbf{R}$, operating on the various spins, that preserves the form of the Hamiltonian. Then a ground state $|\{\sigma_i\}\rangle$ may give rise to another ground state:

$$\mathbf{R}\,|\{\sigma_i\}\rangle = |\{\mathbf{R}\sigma_i\}\rangle \tag{4}$$

The Ising model has an up-down symmetry. If the Hamiltonian is invariant under:

$$\mathbf{R}\sigma_i = -\sigma_i \tag{5}$$

so we have:

$$\hat{\mathcal{H}}(\mathbf{R}\sigma) = -\sum_i J(-\sigma_i)(-\sigma_j) = -\sum_i J\sigma_i\sigma_j. \tag{6}$$

- **order parameter:** By order parameter we mean an average of a measurable microscopic quantity that exhibits singular behavior in the transition point.

  For a magnetic transition it is the magnetization:

  $$m = \langle\sigma\rangle \tag{7}$$

  where the average should be over configurations or location or both - these are equivalent if we believe ergodicity applies, and the system is uniform.

- 'Impossibility' of symmetry breaking. The order parameter m transforms under the symmetry group of the system just like S:

$$\mathbf{R}m = -m \tag{8}$$

This immediately excludes the possibility of broken symmetry:

$$m = \langle\sigma\rangle = \frac{1}{Z}\sum_{\{\sigma_i\}} \sigma_0 e^{-\hat{\mathcal{H}}(\{\sigma_i\}/T} \tag{9}$$

But also:

$$m = \frac{1}{Z}\sum_{\{\mathbf{R}\sigma\}} \sigma_0 e^{-\hat{\mathcal{H}}(\{\mathbf{R}\sigma_i\}/T} = \frac{1}{Z}\sum_{\{\mathbf{R}\sigma\}} (-1)\mathbf{R}\sigma_0 e^{-\hat{\mathcal{H}}(\{\mathbf{R}\sigma_i\}/T} = -m. \tag{10}$$

Since $m = -m$, we must have:

$$m = 0, \tag{11}$$

contrary to our understanding of the ordered phase.

Resolution of this apparent paradox - ergodicity breaking. In a large system it'll take an exponentially (in system size) large time to flip between the various symmetry broken states.

- Broken symmetry from a symmetry-breaking field.

Put a term in the Hamiltonian that breaks the symmetry:

$$\hat{\mathcal{H}}_{SB} = -\sum_i h\sigma_i \tag{12}$$

This term makes the spins point in the + direction when h is positive, and it naturally chooses the ground state of the system - it lifts the degeneracy. We suspect that the symmetry is broken in the infinite system. Therefore a description of a broken symmetry is

$$\lim_{h\to 0}\lim_{N\to\infty} m = m_0. \tag{13}$$

How is that different than what we had before? It is an issue of order of limits. The sum before was implicitly for a finite system, since it assumed ergodicity.What is showed is that:

$$\lim_{N\to\infty}\lim_{h\to 0} m = m_0. \tag{14}$$

C. Length scale - correlation length

The singularity in the magnetization as a function of the magnetic field seems to only be obtained if the system is infinite. A system is large enough to exhibit the sharp behavior associated with critical behavior if its size, L obeys $L > \xi$.

xi is the correlation length, which diverges at the transition point:

$$\xi \sim \frac{1}{|t|^\nu} \tag{15}$$

where $t = (T - T_c)/T_c$.

ξ has an even deeper meaning - it is the correlation length. The rigorous statement associated with the claim above is that the reduced-correlation function decays as:

$$\langle(\sigma_i - \langle\sigma_i\rangle)(\sigma_j - \langle\sigma_j\rangle)\rangle \sim e^{-|i-j|/\xi} \tag{16}$$

But more important even, is the qualitative meaning. A spin σ_i, is sensitive to spins in a sphere of size ξ around it. In order to figure out what it wants to do, it literally 'seeks the advise' of its 'sphere of interest' spins.

A second qualitative viewpoint on the correlation length, is that it determines the size of 'droplets' that point in the same direction. For instance, above the transition, $T > T_c$, even though long range order is not established, still there are some fluctuations in which a group of spins suddenly form a magnetized droplet. The characteristic size of this droplets is ξ.

D. Critical Exponents

In addition to the correlation length diverging, we expect also the following singular behavior:

1. heat capacity:

$$c \sim \frac{1}{|t|^{\alpha}} \tag{17}$$

2. order parameter:

$$m \sim |t|^{\beta} \tag{18}$$

3. susceptibility:

$$\chi \sim \frac{1}{|t|^{\gamma}} \tag{19}$$

4. equation of state at $t = 0$:

$$m \sim H^{1/\delta} \tag{20}$$

5. Correlation length:

$$\xi = \frac{1}{|t|^{\nu}} \tag{21}$$

The exponents α, β, $\gamma\,\delta$ and ν are **universal**.

E. Role of Models - Universality

Systems can be categorized into 'universality classes', which have the same critical behavior (ie critical singularities and exponents), but they differ in the microscopic details of the phases on the two sides of the transition, and in the transition point itself. Systems in the same universality class, share

1. symmetry,
2. dimensionality
3. range of interaction (short vs. long)

All other details are **irrelevant**.