Week 7 - Epsilon expansion

I. WORKING WITH THE PATH INTEGRAL

In order to get some practice before we plunge into the Gaussian approximation let us carry out two quick calculations using the complete path integral formulation: Correlations, and specific heat.

First, an important integral:

$$\int dz d\overline{z} e^{-g|z|^2} = 2\int dx \int dy e^{-g(x^2+y^2)} = \frac{2\pi}{g} \tag{1}$$

another important integral:

$$\int dz d\overline{z} |z|^2 e^{-g|z|^2} = \frac{2\pi}{g^2} \tag{2}$$

All that we do here follows from these two integrals.

A. Correlations from the path integral

The correlation function is:

$$G(\vec{x}_1 - \vec{x}_2) = \langle m(\vec{x}_1) m(\vec{x}_2) \rangle \tag{3}$$

The angled brackets are an average over realizations, which we need to do using the path integral formulation:

$$G(\vec{x}_1 - \vec{x}_2) = \frac{1}{Z}\int D[m] m(\vec{x}_1) m(\vec{x}_2) e^{-\beta F_L[m(x)]} \tag{4}$$

With

$$F_L[m(x)] = \int d^d x \left[\frac{1}{2}(\nabla m)^2 + \frac{1}{2} r m^2 + \frac{1}{4} u m^4\right] \tag{5}$$

This expression looks very opaque, but it can be very easily simplified by using Fourier integrals. I will use the convention:

$$m(\vec{x}) = \int \frac{d^d k}{(2\pi)^d} m_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} \tag{6}$$

and

$$m(\vec{k}) = \int d^d x m(\vec{x}) e^{-i\vec{k}\cdot\vec{x}} \tag{7}$$

In this notation, the Dirac delta function is:

$$\delta^{(d)}(\vec{x}) = \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k}\cdot\vec{x}}. \tag{8}$$

Let's simplify the expression above in stages, also, we assume the Gaussian approximation, in which we neglect $u \to 0$.

First, the free energy:

$$F_L[m(x)] \approx \int d^d x \left[\frac{1}{2}(\nabla m)^2 + \frac{1}{2} r m^2\right] = \int \frac{d^d k}{(2\pi)^d} \frac{1}{2}\left(k^2 + r\right) |m_{\vec{k}}|^2. \tag{9}$$

Now m_k's only appear squared and don't multiply eachother. We just managed to diagonalize the quadratic form that used to be a complicated product - for that we used translational invariance.

Now, there is the product of $m(\vec{x}_1)m(\vec{x}_2)$. Which we can also write as a fourier transform:

$$m(\vec{x}_1)m(\vec{x}_2) = \int \frac{d^d k_1}{(2\pi)^d} \int \frac{d^d k_2}{(2\pi)^d} m_{\vec{k}_1} \overline{m}_{\vec{k}_2} e^{i\vec{k_1}\cdot\vec{x_1} - i\vec{k_2}\cdot\vec{x_2}} \tag{10}$$

where we used the idenetity: $m_{-k} = \overline{m}_k$. This is the result of requiring that m be real.

We want to put this in the expectation value, but we see that there won't be any non-zero expectation value unless we are multiplying the m_k with the m_{-k}. This allows the simplification:

$$G(\vec{x}_1 - \vec{x}_2) = \frac{1}{Z} \frac{d^d k}{(2\pi)^d} \int \frac{d^d k}{(2\pi)^d} \int D[m] |m_{\vec{k}}|^2 e^{i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2)} e^{-\beta F_L[m(x)]} \tag{11}$$

This is easy, because for each k there is only this integral to calculate:

$$\frac{\int dm_{\vec{k}} dm_{-\vec{k}} |m_{\vec{k}}|^2 e^{-\beta \frac{d^d k}{(2\pi)^d}(k^2+r)|m_{\vec{k}}|^2}}{\int dm_{\vec{k}} dm_{-\vec{k}} e^{-\beta \frac{d^d k}{(2\pi)^d}(k^2+r)|m_{\vec{k}}|^2}} \tag{12}$$

but this is simple:

$$\frac{\int dz d\overline{z} e^{-g|z|^2}}{\int dz d\overline{z} e^{-g|z|^2}} = \frac{1}{g} \tag{13}$$

So the above integral is just:

$$\frac{\int dm_{\vec{k}} dm_{-\vec{k}} |m_{\vec{k}}|^2 e^{-\beta \frac{d^d k}{(2\pi)^d}(k^2+r)|m_{\vec{k}}|^2}}{\int dm_{\vec{k}} dm_{-\vec{k}} e^{-\beta \frac{d^d k}{(2\pi)^d}(k^2+r)|m_{\vec{k}}|^2}} = \frac{(2\pi)^d}{d^d k} \frac{1}{r + k^2} \tag{14}$$

the extra piece of the measure sticking out cancels the extra measure we had before. In principle, we could go back and write:

$$\langle m_{\vec{k}_1} \overline{m}_{\vec{k}_2} \rangle = (2\pi)^d \delta^{(d)}(\vec{k}_1 - \vec{k}_2) \frac{1}{r + k^2} \tag{15}$$

And the correlation is given by:

$$G(\vec{x}_1 - \vec{x}_2) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{r + k^2} e^{i\vec{k}(\vec{x}_1 - \vec{x}_2)} \tag{16}$$

when $r > 0$, in the disordered phase, we can find this quickly using the method of residues:

$$G(x) = \frac{\pi}{\sqrt{r}} e^{-|x|\sqrt{r}}. \tag{17}$$

B. (Skipped in class) Singularity in the heat capacity

The heat capacity is a bit more tricky. Here we need a knowledge of the partition function to some extent. The partition function is given as the path integral:

$$Z = J \int D[m_k] \exp \int \frac{d^d k}{(2\pi)^d} \frac{1}{2} \left(k^2 + r\right) |m_{\vec{k}}|^2 \tag{18}$$

where we use the Gaussian approximation, and write J as a jacobian that transfers us from real space to Fourier space integrals. Upto a constant this becomes:

$$\begin{aligned} &\to J' \prod_k \sqrt{\tfrac{2\pi}{k^2+r}} \to J'' \exp\left(\tfrac{1}{2} \sum_k \ln \tfrac{2\pi}{k^2+r}\right) \\ &= J'' \exp\left(\tfrac{1}{2} V \int \tfrac{d^d k}{(2\pi)^d} \ln \tfrac{2\pi}{k^2+r}\right) \end{aligned} \tag{19}$$

but now, clearly we can't integrate over all k's with $k \to \infty$. We need to stop somewhere. Where do we stop? when the wave number is reflecting the coarse graining length of the lattice:

$$|k| < \frac{\pi}{a} = \Lambda \tag{20}$$

Λ is called the cutoff of the theory. In high energy physics, it is the UV cutoff.

The free energy is just the log of the above. So we have:

$$F = const + T\frac{1}{2}V\int^{\Lambda} \frac{d^d k}{(2\pi)^d} \ln\frac{2\pi}{k^2+r} \tag{21}$$

Now, we are looking for the fastest divergence of the heat capacity:

$$C = -T\frac{\partial^2 F}{\partial T^2} \tag{22}$$

but the fastest divergence would come from differentiating the T that is hiding in the $r \propto T - T_C$. Hence:

$$C \sim -\frac{\partial^2}{\partial r^2}\frac{1}{2}V\int^{\Lambda} \frac{d^d k}{(2\pi)^d} \ln\frac{2\pi}{k^2+r} = \frac{1}{2}V\int^{\Lambda} \frac{d^d k}{(2\pi)^d}\frac{1}{(k^2+r)^2} \tag{23}$$

now we rescale k to get rid of the r dependence:

$$k = \sqrt{r}x \tag{24}$$

and obtain:

$$\to \frac{1}{2}V\frac{1}{r^{2-d/2}}\int^{\Lambda/\sqrt{r}} x^{d-1}dx\frac{1}{(x^2+1)^2} \tag{25}$$

this integral is well defined as $x \to 0$, but it is really infinity that we are concerned about. so we ged rid of the 1 in the denominator, and we get:

$$\to \frac{1}{2}V\frac{1}{r^{2-d/2}}\int^{\Lambda/\sqrt{r}} \frac{1}{x^{5-d}}dx \tag{26}$$

this integral diverges if $5 - d \leq 1$. or:

$$d \geq 4 \tag{27}$$

at that point we need to do the integral and keep the parts neer the higher cutoff:

$$\sim \frac{1}{r^{2-d/2}}(\sqrt{r})^{4-d} \sim 1 \tag{28}$$

no divergence. This is just the saddle point result. But now, when $d < 4$, again different things happen, the integral converges, it is just a number, and we get:

$$c \sim \frac{1}{r^{2-d/2}} \tag{29}$$

this means that the critical exponent is just:

$$\alpha = 2 - d/2 = 2 - \nu d \tag{30}$$

consistent with the our mean-field other gaussian exponents.

II. MOMENTUM SHELL RG OF THE ISING LANDAU FREE ENERGY

A. Orientation

We are about to do a $d = 4 - \epsilon$. We expect that when the dimensionality is close to 4 the stable fixed point is really close to our beloved Gaussian fixed point. In fact, we expect a fixed u of order ϵ.

The RG essentially has two steps:

1. Coarse grain the lattice and define a new lattice constant $a' > a$.
2. rescale $a' \to a$, i.e., make a' the new unit of length. This gives rise to the rule $x' = x/b$. x was measure in units of a, and x' is measured in units of a', which is now the new unit of length.

The cutoff momentum is $\Lambda = \frac{1}{a}$. When we coarse grain, we end up with a theory whose cutoff is:

$$\Lambda' = \frac{1}{a'} = \frac{1}{ba} = \frac{\Lambda}{b} \tag{31}$$

Now we rescale the theory - we change our unit of measurement from a to a', this restores the cutoff to its original cutoff:

$$\Lambda' \to \Lambda \tag{32}$$

This is why rescaling is so crucial - we want to end up not just with a Landau free energy with the same sort of terms, but also, with the same cutoff. Only then can we compare the theories before and after RG.

We will see that the decimation step will contribute corrections that are of order u down relative to the contributionts to the RG transformation due to rescaling. We are concerned with u of the order ϵ.

In addition we will concentrate on b that is infinitesimally different from 1:

$$b = e^{d\ell} = 1 + d\ell. \tag{33}$$

This will allow us to obtain RG equations in differential form.

B. Coarse graining

Coarse graining proceeds by integrating over all Fourier components of m_k with $|k| > \Lambda - d\Lambda = \Lambda(1 - d\ell)$.

Define: $m_k^<$ as m_k with $k < \Lambda/b$, and similarly $m_k^>$ as m_k with $\Lambda/b < k < \Lambda$.

The partition function is given by:

$$Z = \int D[m_k] e^{-\int d^dk \frac{1}{2}(k^2+r)|m_k|^2} e^{-\int d^dx m^4 \frac{u}{4}} \tag{34}$$

and now we split our integrals into integrals over the slow m's and the fast m's:

$$\to \int D[m_k^<] e^{-\int_< d^dk \frac{1}{2}(k^2+r)|m_k^<|^2} \int D[m_k^>] e^{-\int_> d^dk \frac{1}{2}(k^2+r)|m_k^>|^2} \tag{35}$$

now, for the RG, we are going to only integrate over the $>$ components.

The next step is to write the fourth-order term as a Fourier integral:

$$\int d^dx m(x)^4 = \int \prod_{i=1}^{d} d^dk_i e^{ik_1x+ik_2x+ik_3x+ik_4x} m_{k_1} m_{k_2} m_{k_3} m_{k_4} \tag{36}$$

But the integral over space, is just a delta function. And we get:

$$\int \prod_{i=1}^{d} d^dk_i \delta^{(d)}(k_1 + k_2 + k_3 + k_4) m_{k_1} m_{k_2} m_{k_3} m_{k_4}. \tag{37}$$

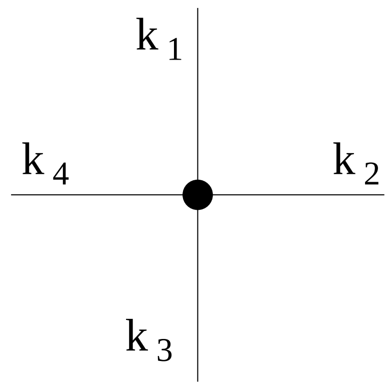


FIG. 1:

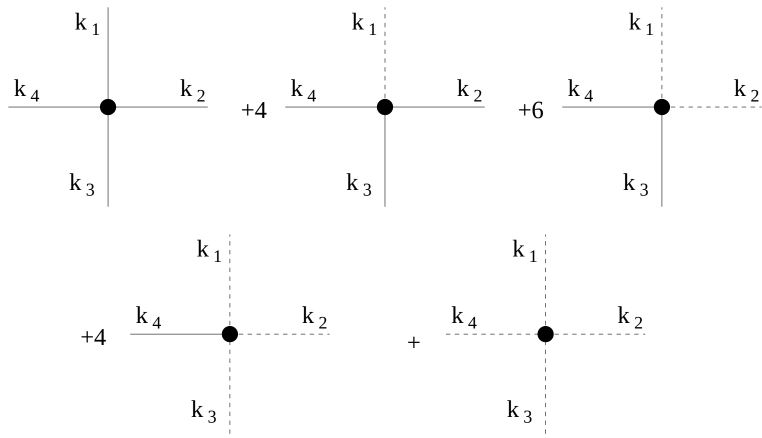


FIG. 2:

C. Feynman diagrams

Now we need to divide this tough integral into smaller-than and bigger-than parts. It is profitable to resort to diagrams. The diagrammatic representation of the um^4 term is something like this:

- the $\frac{u}{4}\delta$ function we represent by a point. Each m we represent by a line. (Fig 1)
- All the m's are equivalent, and each one could be either bigger or smaller then. We could designate the bigger than by a dashed line. This allows five possibilities, with relative degeneracies (Fig 2).

D. Cumulant expansion

The next step is to expand in power's of u. This requires a bit of preparation. Define:

$$V = \frac{u}{4}\int\prod_{i=1}^{d} d^d k_i \delta^{(d)}(k_1+k_2+k_3+k_4) m_{k_1} m_{k_2} m_{k_3} m_{k_4} \tag{38}$$

The partition function we can now write as:

$$Z = \int D[m^<] e^{-\int_< d^d k \frac{1}{2}(k^2+r)|m_k^<|^2 - V^<} \int D[m_k^>] e^{-\int_> d^d k \frac{1}{2}(k^2+r)|m_k^>|^2} e^{-V^>} \tag{39}$$

let's ignore the first piece, since we don't really want to integrate over the slow, low k, modes, and concentrate on the second part:

$$\int D[m_k^>] e^{-\int_> d^d k \frac{1}{2}(k^2+r)|m_k^>|^2} e^{-V^>} = \int D[m_k^>] e^{-\int_> d^d k \frac{1}{2}(k^2+r)|m_k^>|^2} \langle e^{-V^>}\rangle \tag{40}$$

where the averaging is done with respect to the quadratic part of the action.

Let's calculate this average. First step is: Expand the integral:

$$\langle e^{-V^>}\rangle = 1 - \langle V^>\rangle + \frac{1}{2}\langle (V^>)^2\rangle - \frac{1}{3!}\langle (V^>)^3\rangle + \dots \tag{41}$$

The thing to do now is to re-exponentiate the expansion. We start with the first order:

$$1 - \langle V^>\rangle + \frac{1}{2}\langle (V^>)^2\rangle - \frac{1}{3!}\langle (V^>)^3\rangle + \dots = e^{-\langle V^>\rangle} + \frac{\langle (V^>)^2\rangle}{2} - \frac{\langle V^>\rangle^2}{2} - \left(\frac{\langle (V^>)^3\rangle}{3!} - \frac{\langle V^>\rangle^3}{3!}\right) \tag{42}$$

where we corrected the missing higher orders of the $\langle V\rangle$. The same we can do with the second order stuff:

$$= e^{-\langle V^>\rangle + \frac{\langle (V^>)^2\rangle}{2} - \frac{\langle V^>\rangle^2}{2}} - \left(\frac{\langle (V^>)^2\rangle}{2} - \frac{\langle V^>\rangle^2}{2}\right)(e^{-\langle V^>\rangle} - 1) - \frac{\langle (V^>)^3\rangle}{3!} + \frac{\langle V^>\rangle^3}{3!} + \dots \tag{43}$$

but sticking to only third order terms this is:

$$= e^{-\langle V^>\rangle + \frac{\langle (V^>)^2\rangle}{2} - \frac{\langle V^>\rangle^2}{2}} - \frac{\langle (V^>)^3\rangle}{3!} + \frac{1}{2}\langle (V^>)^2\rangle\langle V^>\rangle - \frac{1}{3}\langle V^>\rangle^3 + \dots \tag{44}$$

which we can re-exponentiate:

$$= e^{-\langle V^>\rangle + \frac{\langle (V^>)^2\rangle}{2} - \frac{\langle V^>\rangle^2}{2} - \left(\frac{\langle (V^>)^3\rangle}{3!} - \frac{1}{2}\langle (V^>)^2\rangle\langle V^>\rangle + \frac{1}{3}\langle V^>\rangle^3\right) + \mathcal{O}(V^4)} \tag{45}$$

What you see in the exponent is called the 'cumulant expansion' of $V^>$.

E. Calculation

To wrap up the decimation stage, all that is left is to calculate these contributions. In $\langle V^>\rangle$ the $m^>$ are integrated over. This means that every pair of 'bigger-than' m's can be contracted, and replaced by:

$$\langle m^>_{k_1} m^>_{k_2}\rangle = \delta^{(d)}(k_1 + k_2)\frac{1}{r + k^2} \tag{46}$$

this is represented by making two dashed lines connect and form a loop. .

The simplest term in $\langle V\rangle$ is from the one $m^>$:

$$\langle \frac{u}{4}\int \prod_{i=1}^{4} d^d k_i \delta^{(d)}(k_1+k_2+k_3+k_4) m^>_{k_1} m^<_{k_2} m^<_{k_3} m^<_{k_4}\rangle = \frac{u}{4}\int \prod_{i=1}^{d} d^d k_i \delta^{(d)}(k_1+k_2+k_3+k_4)\langle m^>_{k_1}\rangle m^<_{k_2} m^<_{k_3} m^<_{k_4} = 0 \tag{47}$$

A similar fate befalls the term with three $m^>$.

The most interesting term is the one with two $m^>$:

$$\langle m^>_{k_1} m^>_{k_2} m^<_{k_3} m^<_{k_4}\rangle = \langle m^>_{k_1} m^>_{k_2}\rangle m^<_{k_3} m^<_{k_4} = \delta^{(d)}(k_1 + k_2)\frac{1}{k_1^2 + r} m^<_{k_3} m^<_{k_4} \tag{48}$$

This has a diagrammatic representation in Fig 3. when we do the k integrals and get rid of the δ functions, the contracted line becomes:

$$\int^> d^d k_1 \frac{1}{r + k^2} \approx \frac{1}{8\pi^2}(\Lambda^2) d\ln\Lambda \tag{49}$$

And once all the sums are done this is:

$$3\frac{u}{16\pi^2}\int d^d k |m_k|^2 \Omega_d (\Lambda^2) d\ln\Lambda \tag{50}$$

The $d\ln\Lambda$. shows up in the following way:

$$(\Lambda - d\Lambda)b = \Lambda \tag{51}$$

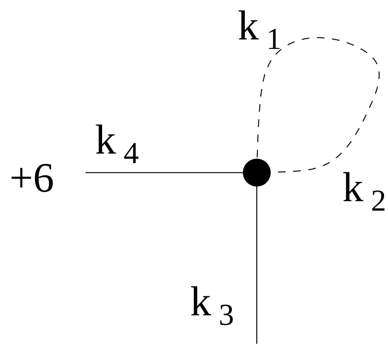


FIG. 3:

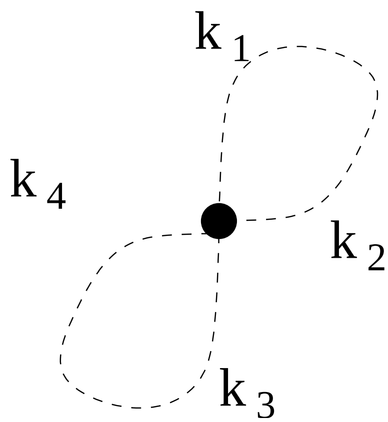


FIG. 4:

or:

$$b = 1 + d\ln\Lambda = 1 + d\ell \tag{52}$$

so $d\ell = d\ln\Lambda$. This is the differential that represents the change of scale in the problem.

When we put this back in the exponent where it belongs, we see that we got an extra mass term:

$$r \to r + \frac{3}{16\pi^2}(\Lambda^2 - r)d\ell \tag{53}$$

Another interesting term is the four $m^>$. This are:

$$\langle m^>_{k_1} m^>_{k_2} m^>_{k_3} m^>_{k_4}\rangle = \langle m^>_{k_1} m^>_{k_2}\rangle\langle m^>_{k_3} m^>_{k_4}\rangle + \langle m^>_{k_1} m^>_{k_3}\rangle\langle m^>_{k_2} m^>_{k_4}\rangle + \langle m^>_{k_1} m^>_{k_4}\rangle\langle m^>_{k_2} m^>_{k_3}\rangle \tag{54}$$

where the three terms represent the three possibilities of putting lines together (Fig. 4).

In this equation we actually neglected the possibility that all m_k represent the same k. This is allowed since the phase space for that is essentially zero.

Now, the resulting term, whatever it is, has no dependence on the m_k - it is just a constant factor in Z. This can we forget. In that sense we are not necessarily keeping Z the same, but we remove a non-singular contribution which is essentially independent of the temperature.

III. MOMENTUM SHELL RG - SECOND ORDER AND RESULTS

A. Second order and Wick's theorem

Notice that in order to calculate the first order what we did was to contract dashed lines. when we look at the second order term, we have two u-terms. Each with all possiblities. The first thing we need to realize - only connected diagrams contribute. How do we know? look at the commulant expansion, the disconnected diagrams contribute

$$\langle V\rangle\langle V\rangle \tag{55}$$

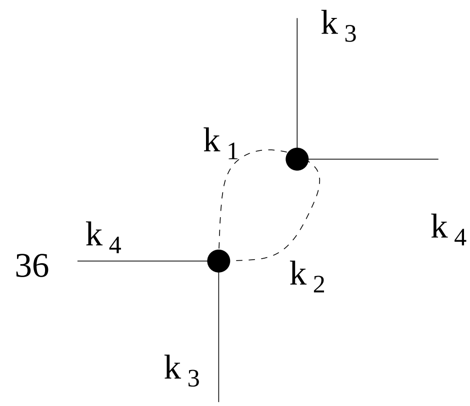
anyway. This would cancel the square of the average. We can do the same argument for the higher order commulant expansion, and in each order you'll see that only fully connected diagrams contribute. This is Wick's theorem.

Let's stick with the second order though: What connected diagrams could we have?

- one dashed line in each - that'll leave six solid lines - m^6 term. This term is not really important for us unless we suspect we have a first order transition on our hands.
- how about four in each? That's just a constant.
- three in each? This is an important term, but it'll be a u^2 contribution to r.
- same thing for 2 dashed lines by 4 dashed lines - u^2 contribution to r.

  Now note that all these guys have two loops. We neglect them at this order. The order in which we are working in is the one-loop order.
- 2 dashed lines by 2 dashed lines is the important piece. It is a u^2 renormalization of u. We will now calculate it.

The diagram we want to calculate is this:



Now, we can separate this diagram into two parts, the solid and the dashed:

$$2\frac{36}{2}\frac{u^2}{16}\int\prod_{i=1}^{4}d^dk_i\delta^{(d)}(k_1+k_2+k_3+k_4)m^<_{k_1}m^<_{k_2}m^<_{k_3}m^<_{k_4}\int^> d^dk_5d^dk_6\frac{1}{k_5^2\cdot k_6^2}\delta^{(d)}(k_1+k_2+k_5+k_6) \tag{56}$$

but here we can simplify things - we assume that k_1, $k_2 \ll \Lambda$, and can be neglected compared to k_5 and k_6. So to first order we have:

$$\rightarrow 36\frac{u^2}{16}\int\prod_{i=1}^{4}d^dk_i\delta^{(d)}(k_1+k_2+k_3+k_4)m^<_{k_1}m^<_{k_2}m^<_{k_3}m^<_{k_4}\int^> d^dk_5\frac{1}{k_5^4} \tag{57}$$

so the addition to u is:

$$4\cdot 36\frac{u^2}{16}\int^> d^dk_5\frac{1}{k_5^4} \tag{58}$$

Now, we are not really making any error if we assume that the integral is four dimensional. This gives:

$$9u^2\frac{S_4}{(2\pi)^4}d(\ln\Lambda) = 9u^2\frac{1}{8\pi^2}d\ell \tag{59}$$

B. rescaling

Now we can pause for a minute and collect our thoughts. By integrating over fast modes upto order ϵ - i.e. - one loop, we obtained contributions to the low-momentum action. The new action now reads:

$$F_< = \int d^d x \left[\frac{1}{2}(\nabla m^<)^2 + \frac{1}{2}\left(r + \frac{3}{2\pi^2}\frac{u}{4}\Lambda^2 d\ell \right)(m^<)^2 + \frac{1}{4}\left(u - \frac{9}{8\pi^2}u^2 d\ell \right)(m^<)^4 \right] \tag{60}$$

But this is still before rescaling! We need to now carry out the rescaling that brings back $\Lambda - d\Lambda$ to Λ. This step is:

$$x' = x/b = x(1 - d\ell) \tag{61}$$

Tp keep the gradient term constant we still need to rescale m in the same way. This implies that still:

$$\eta = 0 \tag{62}$$

and:

$$m' = mb^{d-2} \tag{63}$$

Note that this is because we did not generate any gradient terms to order ϵ. So all we know really is that:

$$\eta = O(\epsilon^2) \tag{64}$$

which is indeed true, but too complicated for us right now.

Now, the rescaling of x and m is already enough to find the rescaling of all the other terms in the action:

$$r' = rb^2 = r(1 + 2d\ell) \;\; u' = ub^{4-d} = u(1 + \epsilon d\ell) \tag{65}$$

The contributions in these brackets just add up to the contributions from the decimation procedure. Note that the rescaling is a correction of order $d\ell$ and therefore if we rescale the decimation correction we get $d\ell^2$ which we set to zero.

C. RG flow equations

All together this gives:

$$\begin{array}{c} \frac{dr}{dl} = 2r + \frac{3(\Lambda^2 - r)}{8\pi^2}u \\ \frac{du}{dl} = \epsilon u - \frac{9}{8\pi^2}u^2 \end{array} \tag{66}$$

that's it! Now it is time for analysis. There is a fixed point at $r = u = 0$ indeed. Our old friend - the gaussian fixed point. If we linearize the RG transformation near it, we have:

$$\frac{d}{d\ell}\begin{pmatrix} r \\ u \end{pmatrix} = \begin{pmatrix} 2 & \frac{3\Lambda^2}{8\pi^2} \\ 0 & \epsilon \end{pmatrix}\begin{pmatrix} r \\ u \end{pmatrix} \tag{67}$$

which indeed has two positive eigenvalues:

$$\lambda_1 = 2 \;\; \lambda_2 = \epsilon \tag{68}$$

the only thing new here is the direction - the eigenvalue that corresponds to ϵ is turning to the negative-r direction.

But instead of analysing the unstable gaussian fixed point, let's move to the critical fixed point. There is another solution that will give zero for the flow. From the u flow we have:

$$u^* = \epsilon\frac{8\pi^2}{9} \tag{69}$$

which, upon solving the first equation gives:

$$r^* = -\frac{3\Lambda^2}{16\pi^2}\epsilon\frac{8\pi^2}{9} = -\epsilon\frac{2}{3}\Lambda^2 \tag{70}$$

This is called the "Wilson-Fisher fixed point".

We are not done before linearizing the equations and finding its eigenvectors, and scaling dimensions - i.e. - the eigenvalues of the transformation. To do this we differentiate at the fixed point and get:

$$\frac{d}{d\ell}\begin{pmatrix}\delta r\\ \delta u\end{pmatrix} = \begin{pmatrix}2 & \frac{3\Lambda^2}{8\pi^2}\\ 0 & \epsilon - \frac{9}{4\pi^2}u^*\end{pmatrix}\begin{pmatrix}\delta r\\ \delta u\end{pmatrix} \tag{71}$$

The flow matrix is then:

$$\begin{pmatrix}2 & \frac{3\Lambda^2}{8\pi^2}\\ 0 & -\epsilon\end{pmatrix} \tag{72}$$

Now the eigenvalues are:

$$\lambda_1 = 2 \;\; \lambda_2 = -\epsilon \tag{73}$$

Indeed - unstable only in the r direction. If we take into account also the r in the $r + k^2$, which I neglected before for simplicity, we get:

$$\lambda_1 = 2 - \frac{3}{8\pi^2}u^* = 2 - \frac{1}{3}\epsilon \tag{74}$$

this is the critical fixed point we were looking for! If you finish solving for the eigen vectors, you'll see the same relevant r direction, and a slight change in this direction, which discloses the critical manifold.

The r scaling dimension is $1/\nu$, which means, to order ϵ:

$$\nu = \frac{1}{2} + \frac{\epsilon}{12} \tag{75}$$

So we even got the correction for ν. Using our universal relationships between the critical exponents, we get the corrected greek letters:

$$\begin{array}{c}\alpha = 2 - \nu d = 2 - \frac{\epsilon}{3} + \frac{\epsilon}{2} = \frac{\epsilon}{6}\\ \gamma = (2-\eta)\nu = 1 + \frac{\epsilon}{6}\\ \beta = \frac{1}{2}(2 - \alpha - \gamma) = \frac{1}{2} - \frac{\epsilon}{6}\\ \delta = 1 + \frac{\gamma}{\beta} = 3 + \epsilon\end{array} \tag{76}$$

and these are true to order ϵ.