due date: April 21th, 5pm.

Ph 127c - Problem set 1

1. Superfluids as X-Y magnets.

   The superfluid transition at dimensions $d > 2$, is a second order phase transition that breaks $U(1)$ symmetry. The minimal Landau free energy that describes this transition is:

   $$\mathcal{F}[\psi] = \int d^d x \left[\frac{1}{2}|\nabla\psi|^2 + \frac{1}{2}r|\psi|^2 + \frac{1}{4}u|\psi|^4 \right] \tag{1}$$

   with $r \propto T - T_c$.

   (a) In two-dimensions, there is no phase transition at T_c, although the Landau free energy is still correct. Deep in the superfluid phase, $T \ll T_c$, show that the free energy has the term:

   $$F = \int d^d r \frac{1}{2}\chi\nabla\phi^2 \tag{2}$$

   where ϕ is the phase of the order parameter and ψ_m its magnitutde, $\psi = \psi_m e^{i\phi}$. What is the free energy functional for ψ_m? what is χ at the mean field in terms of the parameters of the Landau theory?

   (b) Estimate the fluctuations in the magnitude of the order parameter, i.e., $\langle(\psi_m - \langle\psi_m\rangle)^2\rangle$, neglecting any spatial fluctuations. Justify using only Eq.(2) as the free energy for $T \ll T_c$

   (c) The free energy, Eq. (2), is only valid at large wavelengths. To see this, put $\psi = |\psi|e^{ikx}$. How big can the gradient of ϕ be before the superfluid disappears? Assume that you can neglect fluctuations around mean field. What is this gradient in terms of the correlation length of the Landau theory of Eq. (1)?

2. QLRO. Consider a non-homogeneous x-y magnet in two dimension, which has different stiffnesses in the x and y direction:

   $$F = \int d^2 r \left[\frac{1}{2}\chi_x \left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{1}{2}\chi_y \left(\frac{\partial\phi}{\partial y}\right)^2 \right] \tag{3}$$

   Assume also that this continuum theory is only valid upto momenta $|k| < \Lambda$.

   (a) In an x-y magnet consisting of individual magnetons, what is Λ measuring?

   (b) Find the correlation function:

   $$C(\vec{r}) = \langle e^{i\phi(\vec{r}) - i\phi(0)} \rangle \tag{4}$$

   In an infinite system at temperature T, and with $|\vec{r}| \gg \Lambda^{-1}$. You're allowed to use the approximation:

   $$1 - \cos\vec{k}\cdot\vec{r} \approx \begin{cases} 0 & |\vec{k}\cdot\vec{r}| < 1 \\ 1 & |\vec{k}\cdot\vec{r}| > 1. \end{cases}$$

3. A Josephson junction in a circuit.

   Consider a system consisting of two Superfluids, A and B, of the same particles, which can exchange particles between them through a Josephson junction. The Hamiltonian describing this system is:

   $$\hat{\mathcal{H}} = \frac{1}{2C_A}\hat{N}_A^2 - \mu_A \hat{N}_A + \frac{1}{2C_B}\hat{N}_B^2 - \mu_B \hat{N}_B - J\cos\left(\hat{\phi}_A - \hat{\phi}_B\right) \tag{5}$$

   (a) What are the (Ehrenfest) equations of motion for $\langle\hat{N}_{A,B}\rangle$, $\langle\hat{\phi}_{A,B}\rangle$? What is the static solution for them?

   (b) Starting with the two condensates at the ground state of this Hamiltonian (i.e., the static solution found above), Condesate A is lowerd such that $\mu_A \to \mu_A + V$. Find the motion that ensues. In your solution, assume that $\langle\hat{\phi}_A - \hat{\phi}_B\rangle \ll 1$. What conditions on $C_{A,B}, V, J$ has to be fulfilled for this to be true?

   (c) In general, what is the relationship between the potential energy difference, ΔV, between the two condensates, and the current flowing through the Josephson junction? Show that $\Delta V = L\frac{dI}{dt}$; what is the inductance L?

4. Fourier transform of $\phi(\vec{r})$ describing a vortex. In class we showed that a vortex is described by $\phi = \theta$.

   (a) What is the Fourier transform of this function? Hint: A simple way to calculate this FT is by using the transformed Poisson equation, equating $\nabla^2\phi$ to the Laplacian arising for a branch cut where ϕ jumps from 0 to 2π along the line $x = 0$, $y < 0$.

   (b) * Find using the result in 4a the energy of a vortex, for a superfluid with stiffness χ and coherence length ξ in a system of radius R_{max}. Make sure that the result agrees with the one obtained in class.