due date: April 21th, 5pm.

Ph 127c - Problem set 1 Solutions

1. Superfluids as X-Y magnets.

   (a) If we separate the order parameter ψ to phase ϕ and magnitude $\psi_m = |\psi|$, i.e. $\psi = \psi_m e^{i\phi}$, we can easily substitute this representation into the free energy and get:

$$F = \int d^d x \frac{1}{2}\psi_m^2 (\nabla\phi)^2 + \int d^d x \left(\frac{1}{2}(\nabla\psi_m)^2 + \frac{r}{2}\psi_m^2 + \frac{u}{4}\psi_m^4\right) \quad (1)$$

   Indeed we get the term with the phase gradient, and the stiffness is $\chi = \psi_m^2$. At the mean field level $\psi_m^2 = \frac{|r|}{u}$ for $T < T_c$.

   (b) Neglecting all spatial gradients, we'd like to calculate $\langle(\psi_m - \langle\psi_m\rangle)^2\rangle = \langle\psi_m^2\rangle - \langle\psi_m\rangle^2$. To simplify things a bit we expand the action for ψ_m around the mean field solution $\psi_m^2 = \frac{r}{u} \equiv \psi_0^2$, and we get :

$$F \approx \int d^d x(-r)(\psi_m - \psi_0)^2 \quad (2)$$

   deep in the superfluid phase, $T << T_c$, r is large and negative, so we can extend the lower limit of integration to $-\infty$, and we then get:

$$\langle\psi_m^\alpha\rangle = \frac{\int d\psi_m \psi_m^\alpha e^{-\beta|r|(\psi_m-\psi_0)^2}}{\int d\psi_m e^{-\beta|r|(\psi_m-\psi_0)^2}} \quad (3)$$

   which gives a variance of:

$$\langle(\psi_m - \langle\psi_m\rangle)^2\rangle = \frac{T}{2|r|} \quad (4)$$

   so the deeper we are in the superfluid phase, the larger $|r|$ is, and the fluctuation in the magnitude of the order parameter are suppressed by $1/|r|$.

   (c) If we assume the phase of the order parameter is $\phi = ikx$, then the free energy due to the phase fluctuation is:

$$F_\phi = \int d^d r \frac{1}{2}\chi|\nabla\phi|^2 = \int d^d r \frac{1}{2}\psi_m^2 k^2 \quad (5)$$

   this term obviously modifies the free energy of the magnitude of the order parameter ψ_m, namely it shifts the quadratic coefficient r to $\tilde{r} = r + k^2$. When k is large $\tilde{r}$ can become positive, even when r is negative, and take the system out of the superfluid phase. That happens when $k^2 = |r|$.

2. QLRO.

   (a) Whenever there is a lattice in the problem, the short length scale, which is the lattice length a, enters through the momentum cutoff $\Lambda \propto 1/a$.

   (b) Calculation this correlation function closely follows the isotropic calculation done in class. First we write the free energy and the correlation function in k space:

$$F = \frac{1}{2}\int \frac{d^2k}{(2\pi)^2}(\chi_x k_x^2 + \chi_y k_y^2)|\phi_k|^2 \quad (6)$$

   and the correlation function in terms of the fourier components is:

$$C(\vec{r}) = \langle \exp\left(i\int \frac{d^2k}{(2\pi)^2}(e^{i\vec{k}\cdot\vec{r}} - 1)\phi_k\right)\rangle \quad (7)$$

   Writing this correlation function explicitly:

$$C(\vec{r}) = \frac{1}{Z}\int D[\phi]\exp\left(-\frac{1}{2}\frac{1}{T}\int \frac{d^2k}{(2\pi)^2}(\chi_x k_x^2 + \chi_y k_y^2)|\phi_k|^2 + i\int \frac{d^2k}{(2\pi)^2}(e^{i\vec{k}\cdot\vec{r}} - 1)\phi_k\right) \quad (8)$$

We can complete the square for each k component and do the gaussian integral to get:

$$= \exp\left(-\frac{1}{2}\int^{\Lambda}\frac{d^2k}{(2\pi)^2}T\frac{|e^{i\vec{k}\cdot\vec{r}}-1|^2}{(\chi_x k_x^2+\chi_y k_y^2)}\right) \tag{9}$$

Now we may use the approximation:

$$|e^{i\vec{k}\cdot\vec{r}}-1|^2 = 2(1-\cos\vec{k}\cdot\vec{r}) \approx \begin{cases} 0 & |\vec{k}\cdot\vec{r}|<1 \\ 2 & |\vec{k}\cdot\vec{r}|>1. \end{cases}$$

This approximation sets the lower limit of integration to $|\vec{k}| = 1/r$. The correlation function now becomes, in polar coordinates in momentum space:

$$= \exp\left(-\frac{T}{2(2\pi)^2}\int_{1/r}^{\Lambda}\frac{k\,dk}{k^2}\int_0^{2\pi}d\theta\frac{1}{\chi_x\cos(\theta)^2+\chi_y\sin(\theta)^2}\right) \tag{10}$$

the k integral gives $\ln(\Lambda r)$, and the angular integral is a tabulated integral which gives $2\pi/\sqrt{\chi_x\chi_y}$ (you should be careful with branch cuts when looking up this integral in tables, and notice that it recovers the isotropic limit for $\chi_x = \chi_y$). Putting everything together:

$$C(\vec{r}) = \frac{1}{(|\vec{r}|\Lambda|)^{T/2\pi\sqrt{\chi_x\chi_y}}} \tag{11}$$

we see that the anisotropy did not make much of a difference.

3. A Josephson junction in a circuit.

$$\hat{\mathcal{H}} = \frac{1}{2C_A}\hat{N}_A^2 - \mu_A\hat{N}_A + \frac{1}{2C_B}\hat{N}_B^2 - \mu_B\hat{N}_B - J\cos\left(\hat{\phi}_A - \hat{\phi}_B\right) \tag{12}$$

(a) The equation of motion of the phase is:

$$\left\langle\frac{d\phi_A}{dt}\right\rangle = -\frac{i}{\hbar}\langle[\hat{\mathcal{H}},\hat{\phi}_A]\rangle \tag{13}$$

As seen in class, the commutation relation $[\hat{N}_A,\hat{\phi}_A] = i$ allows us to simplify to commutator above to:

$$[\hat{\mathcal{H}},\hat{\phi_A}] = -\frac{1}{i}\frac{\partial\hat{\mathcal{H}}}{\partial N_A} \tag{14}$$

We have the explicit form of the Hamiltonian and we can take the derivatives:

$$\langle\frac{d\phi_j}{dt}\rangle = \frac{1}{\hbar}\left(\frac{\langle N_j\rangle}{C_j} - \mu_j\right) \tag{15}$$

where $j = A, B$. Similarly:

$$\left\langle\frac{dN_A}{dt}\right\rangle = -\frac{i}{\hbar}\langle[\hat{\mathcal{H}},N_A]\rangle = -\frac{i}{\hbar}\left\langle\frac{1}{i}\frac{\partial\hat{\mathcal{H}}}{\partial\hat{\phi}_A}\right\rangle = -\frac{J}{\hbar}\langle\sin(\hat{\phi}_A-\hat{\phi}_B)\rangle \tag{16}$$

And current conservation guarantees that:

$$\left\langle\frac{dN_B}{dt}\right\rangle = -\left\langle\frac{dN_A}{dt}\right\rangle \tag{17}$$

The static solutions are the ones for which the time derivative vanish. The static solution is:

$$\langle N_j\rangle = \mu_j C_j; \quad \Delta\phi = \langle\hat{\phi}_A - \hat{\phi}_B\rangle = 0(mod\,2\pi) \tag{18}$$

(b) Assuming $\Delta\phi = \langle\hat{\phi}_A - \hat{\phi}_B\rangle \ll 1$ we can replace $\sin(\Delta\phi)$ with $\Delta\phi$. Let us also define

$$\Delta N = \left[\frac{\langle N_A\rangle}{C_A} - \mu_A - \left(\frac{\langle N_A\rangle}{C_B} - \mu_B\right)\right] \cdot C_{AB} \tag{19}$$

with $\frac{1}{C_{AB}} \equiv \frac{1}{C_A} + \frac{1}{C_B}$. Then the equations of motion reduce to the coupled equations:

$$\frac{d(\Delta\phi)}{dt} = \frac{1}{\hbar}\frac{1}{C_{AB}}\Delta N \tag{20}$$

$$\frac{d(\Delta N)}{dt} = -\frac{J}{\hbar}\Delta\phi \tag{21}$$

The initial conditions given translate to $\Delta N(t=0) = -V \cdot C_{AB}$ and $\Delta\phi(t=0) = 0$. These two equations are easily solved for all $t > 0$:

$$\Delta N(t) = -V C_{AB}\cos(\omega t) \tag{22}$$

$$\Delta\phi(t) = -\frac{\hbar}{J} V C_{AB}\omega \sin(\omega t) \tag{23}$$

the oscillation frequency is:

$$\omega = \sqrt{\frac{J}{\hbar^2 C_{AB}}} = \sqrt{\frac{J}{\hbar^2}\left(\frac{1}{C_A} + \frac{1}{C_B}\right)} \tag{24}$$

For our approximation to be valid, we need the maximal value of $\Delta\phi$ to satisfy $\Delta\phi_{max} \ll 1$. This will happen when $\frac{\hbar}{J} V C_{AB} w \ll 1$, i.e. when:

$$\frac{V}{\sqrt{J\left(\frac{1}{C_A} + \frac{1}{C_B}\right)}} \ll 1 \tag{25}$$

(c) The current going from A to be B is:

$$I = -\frac{J}{\hbar}\langle\sin(\Delta\phi)\rangle \approx -\frac{J}{\hbar}\langle\Delta\phi\rangle \approx \tag{26}$$

taking a time derivative and using the above solution for $\Delta\phi(t)$ at $t \to 0$:

$$\frac{dI}{dt} = \frac{J}{\hbar}\frac{\hbar}{J} V C_{AB}\omega^2 = \frac{J}{\hbar^2} V \tag{27}$$

So the function acts as an inductor with inductance $L = \hbar^2/J$.

4. Vortex energy in k space.

In this problem we want to calculate the energy of a vortex. This can be done easily in real space, but for practice, let's do it in momentum space. For a given field configuration $\phi(\vec{r})$, the energy is:

$$F = \int d^2x \frac{\rho_s}{2m}(\nabla\phi)^2 = \frac{\rho_s}{2m}\int \frac{d^k}{(2\pi)^2}|\vec{k}\phi_k|^2 \tag{28}$$

We know that the field of a vortex has a branch cut. It is a little easier to deal with the velocity field, the gradient of $\phi(\vec{r})$, since its singularity is just a point singularity. Defining $\vec{v}(\vec{r}) \equiv \nabla\phi(\vec{r})$, the energy can be also written as:

$$F = \frac{\rho_s}{2m}\int \frac{d^k}{(2\pi)^2}|\vec{v}_k|^2 \tag{29}$$

(note: this is not the usual normalization of a the velocity vector). It is quite simple to find $\vec{v}_k$. We know that in real space $\vec{v}$ has a singularity at the origin:

$$\nabla \times \vec{v}(\vec{r}) = 2\pi\delta(\vec{r}) \tag{30}$$

This equation looks even simpler in momentum space:

$$\vec{k} \times \vec{v}_k = 2\pi \tag{31}$$

A solution to this equation is then:

$$\vec{v}_k = \frac{2\pi}{k^2}\,(k_y, -k_x) \tag{32}$$

(another note: we cannot get ϕ_k by simply dividing by k). Now we can plug this into the energy expression, noting that $|\vec{v}_k| = 1/k$, to get:

$$F = 2\pi \frac{1}{(2\pi)^2} \frac{\rho_s}{2m} \int_{1/R_{max}}^{1/\xi} \frac{dk}{k} \frac{(2\pi)^2}{k^2} = \frac{\pi \rho_s}{m} \ln\left(\frac{R_{max}}{\xi}\right) \tag{33}$$

As always, the momentum cutoffs are set by the length natural to the problem, the system size and the core size. This result for the energy agrees with the result obtained in class in real space.