I. VORTEX PARTITION FUNCTION

Finally, we can write $Z_v$ properly:

$$Z_v = \sum_{n=0}^{\infty} \frac{1}{n! \cdot n!} e^{-2nE_c} \int \prod_{i=1}^{2n} D[\vec{x}_i] e^{\frac{2\pi}{\Lambda} \sum_{i>j} \ln |\vec{x}_i - \vec{x}_j|}$$

(1)

Where we also assumed that $i > n$ have $p_i = -1$ and $i \leq n$ have $p_i = 1$. The factorials are there to eliminate double-counting of the vortex configurations. Also $\zeta = e^{-E_c/T}$ is the fugacity of vortices, parametrized in terms of core energy.

II. THE KOSTERLITZ ARGUMENT FOR A VORTEX-UNBINDING PHASE TRANSITION

The analysis of the vortex contribution is rather complex. Before we dive in, we should get a sense for what awaits - a phase transition.

The vortex forms a neutral plasma. If the vortices act as free particles, they will degrade the fragile QLRO into no order at all. Phase-phase correlations are very senetive to the locations of each vortex, and therefore, if we average over vortex location, we’ll get zero correlations.

So is there any hope for the superfluid? There is, if the vortices can pair up into neutral, harmless, entities. What is the chance of the happening? To answer this, we need to find the free energies of pairs, and free plasma.

What is the free energy of a vortex pair? Assuming the vortices comprising the pair are very close ($1/\Lambda$ distance), we can neglect their interaction. On the other hand, they have entropy. Thus:

$$F_{\text{pair}} = -T \ln(R_{\text{max}}^2) = -2T \ln R_{\text{max}}$$

(2)

But if this pair breaks down, it gains the entropy from one particle, but loses interaction:

$$F_{\text{unpaired}} = 2\pi \chi \ln R_{\text{max}} - 2T \ln(R_{\text{max}}^2) = 2\pi \chi \ln R_{\text{max}} - 4T \ln R_{\text{max}}$$

(3)

Now, clearly pairs are allowed to proliferate - they only have entropy (neglecting their core energies). But they can not destroy the QLRO. But what about unbound pairs, which definitely can destroy the QLRO? For vortices to be bound into pairs, the condition is that the Boltzmann factor associated with them vanishes, and hence:

$$F_{\text{unpaired}} = 2\pi \chi \ln R_{\text{max}} - 4T \ln R_{\text{max}} > 0$$

(4)

And therefore when

$$T < \frac{1}{2} \pi \chi.$$  

(5)

In this case single unbound vortices can not propogate, and vortices are bound in to pairs, and the QLRO is preserved. But when:

$$T > \frac{1}{2} \pi \chi,$$

(6)

at higher temeperatures, vortices unbind, and the QLRO is destroyed. The critical temperature is the Kosterlitz-Thouless temperature, $T_{KT} = \frac{1}{2} \pi \chi$.

III. RENORMALIZATION GROUP APPROACH

While we can see from the Kosterlitz argument that a phase transition should indeed occur, we know nothing about its universality class, critical behavior, or anything of the sort. Whatever it may be, there is no symmetry breaking associated with it.
The RG analysis of this transition starts with the partition function:

\[ Z_v = \sum_{n=0}^{\infty} \frac{1}{n!^2} \zeta^{2n} \int \prod_{i=1}^{2n} D[\vec{x}_i] e^{-K \sum_{i>j} p_i p_j \ln |\vec{x}_i - \vec{x}_j|/\Lambda} \]

(7)

Where I put \( K = 2\pi \chi / T \), for shorthand. Each term in this function describes a neutral gas with \( 2n \) vortices. Note that we put in a fugacity \( \zeta = e^{\mu / T} \) up there, to get rid of the unknown core action. What argument can we use to construct an RG analysis?

When we were discussing momentum-shell RG in the precious quarter, we basically devised a way of knocking off pieces from the partition function, which was almost completely resolved in momentum space. The way I have the problem her set up, we can’t really do this. But we can still consider what happens when we change the cutoff.

The cutoff is the smallest distance that vortices can come to each other. If a vortex and antivortex come any closer than, say, \( 1 / \Lambda \), we can just say that they annihilated.

What happens when we raise this cutoff? Suppose the size of vortices goes from \( \Lambda^{-1} \rightarrow (\Lambda')^{-1} = \Lambda^{-1}(1 + d\ell) \). and let’s look at the \( 2n \) vortex term. There are several distinct cases:

- all the differences \( |\vec{x}_i - \vec{x}_j| > \Lambda^{-1}(1 + d\ell) \). In this case increasing \( \Lambda^{-1} \) doesn’t seem to do much since the vortices are sufficiently far apart. Nevertheless, since the cutoff appears inside the log, changing the cutoff will give us rescaling corrections through the interactions.

- Another possibility, which makes the physics clear, is that a vortex and an anti-vortex happen to be so close, that when the core-size is increased, their cores overlap. To keep the consistency of our approach we must say that they then annihilate. This is funny though - instead of looking at \( 2n \) vortices, we are actually looking at \( 2n - 2 \) vortices. But this means that this case contributes to the previous term. This contribution will be proportional to \( d\ell \). Can you say why? It’s proportional to the area of the ring around the core.

\[ \propto 2\pi \frac{1}{\Lambda} d(\Lambda^{-1}) = 2\pi \Lambda^{-2} d\ell \]

(8)

This is equivalent to the decimation correction to the interaction.

- To be thorough, we must also consider the possibility that more than one pair will be in annihilation position. This can be neglected, however, since it is proportional to \( d\ell^2 \).

- To be really thorough we need to consider other cases - what if two vortices of the same orientation are close to eachother? This we can ignore for a different reason: The repulsion between these vortices is so strong, that such a configuration is unlikely by a factor that goes roughly as \( e^{2\pi \chi \ln(\Lambda / T)} / T \).

From these arguments we see that a decimation and rescaling steps can be constructed by increasing core size, and looking at vortex and anti-vortex pairs annihilating into the vacuum; a process that takes probability from the \( 2n \) term, and puts it in the \( 2n - 2 \) term. Next, we calculate the effects of these steps.

### A. Changing \( \Lambda \) and rescaling.

Because it is so much simpler to do rescaling then to calculate interactions corrections, let is ignore the decimation step for starters. To carry out the rescaling, we first increase the core size, which reduces \( \Lambda \):

\[ \Lambda = \Lambda'(1 + d\ell) \]

(9)

This implies:

\[ K \sum_{i>j} p_i p_j \ln |\vec{x}_i - \vec{x}_j|/\Lambda = K \sum_{i>j} p_i p_j \ln |\vec{x}_i - \vec{x}_j|/\Lambda' + K \sum_{i>j} p_i p_j \ln(1 + d\ell) \]

\[ = K \sum_{i>j} p_i p_j \ln |\vec{x}_i - \vec{x}_j|/\Lambda' - nKd\ell \]

(10)

Where the last line results from the fact that for each vortex, there is one more antivortex attracting it, than vortices repelling it:

\[ \sum_{i>j} p_i p_j = \frac{1}{2} \sum_{i,j} (1 - \delta_{i,j})p_i p_j = -\frac{1}{2} \sum_{i=1}^{2n} 1 = -n \]

(11)
Next, we rescale $\vec{x}_i$:

$$\vec{x}_i' = \vec{x}_i\Lambda$$  \hspace{1cm} (12)

which implies:

$$d^2\vec{x}_i = d^2\vec{x}_i' (1 + 2d\ell)$$  \hspace{1cm} (13)

$2d\ell$ due to the two dimensions.

From these two contributions we see:

$$\zeta^{2n}Z_{2n} = \zeta^{2n}(1 + 2d\ell)^{2n}e^{-nKd\ell}Z_{2n}'$$  \hspace{1cm} (14)

But this can be construed as:

$$\zeta^{2n} \rightarrow (e^{\ln \zeta + (2-K/2)d\ell})^{2n}$$

$$\ln \zeta' = \ln \zeta + (2-K/2)d\ell$$  \hspace{1cm} (15)

And finally:

$$\frac{d\zeta}{d\ell} = \left(2 - \frac{K}{2}\right)\zeta = \left(2 - \frac{\pi\chi}{T}\right)$$  \hspace{1cm} (16)

This is a really neat result. Just by rescaling we see that if the temperature is above $T_{KT} = \pi\chi/2$, $\zeta$, which is the fugacity (i.e., density!) of vortices is relevant, and they proliferate. At lower temperatures, vortices are irrelevant, it seems, and the QLRO should be preserved.

In this ‘naive rescaling’ approximation, it seems that $K$ doesn’t flow. But from a close look at the decimation, we will see that this is not the case.

**B. Decimation**

For the decimation contribution, we need to calculate the contribution to the partition function of $2^n$ vortices, from the configurations that have a vortex-anti-vortex pair. This analysis turns out to be quite a bit more difficult then the so-called ‘naive rescaling’ approximation.

Let’s look at the $2^n$ vortices term:

$$\frac{1}{n!}\zeta^{2n} \int \prod_{i=1}^{2(n-1)} d^2\vec{x}_i d\vec{x}_{2n-1} d\vec{x}_{2n} e^{K \sum_{i>j} p_i p_j \ln |\vec{x}_i - \vec{x}_j|}$$  \hspace{1cm} (17)

Where I separated the last two coordinates from the differential product. This way of writing the term, we already chose, say, the $p_i$ with an odd index to be vortices, and $p_i$ with even index to be antivortices (otherwise the factor in front would have been $1/(2n!)$).

Let us now choose deliberately the $2n - 1$ vortex and the $2n$ antivortex, to be close to each other. Since this implies we chose one of the $n$ vortices, and one of the $n$ antivortices, we need to multiply the term by $n^2$, and separate

$$\frac{1}{2} (\vec{x}_{2n} + \vec{x}_{2n-1}) = \vec{x}_{cm} \quad \Delta \vec{x} = \vec{x}_{2n} - \vec{x}_{2n-1} = \frac{1}{\Lambda} \hat{\nu}$$  \hspace{1cm} (18)

Where $\hat{\nu}$ is a unit vector. This leads to:

$$\rightarrow \frac{n^2}{n!} \zeta^{2n} \int \prod_{i=1}^{2(n-1)} d^2\vec{x}_i d^2\vec{x}_{cm} d\vec{\nu} e^{K \sum_{i>j} p_i p_j \ln |\vec{x}_i - \vec{x}_j| - S_{dipole}}$$  \hspace{1cm} (19)

Where:

$$S_{dipole} = -\sum_{i=1}^{2n-2} K p_i \frac{\hat{\nu}}{\Lambda} \cdot \nabla_{\vec{x}_{cm}} \ln |\vec{x}_i - \vec{x}_{cm}| = \sum_{i=1}^{2n-2} K p_i \frac{\hat{\nu}}{\Lambda} \cdot \frac{(\vec{x}_{cm} - \vec{x}_i)}{|\vec{x}_{cm} - \vec{x}_i|^2}.$$  \hspace{1cm} (20)
is the interaction between all the 'unpaired' vortices and the pair $2n - 1$, $2n$, and $\frac{d\ell}{A} d\varphi = d^2 \Delta \vec{x}$.

The dipole-action is inversely proportional to the cutoff times the distance to the dipole, $(\vec{x}_i - \vec{x}_{cm})\Lambda$, and therefore we can expend $e^{-S_{dipole}}$. Consider:

$$\int d^2 \vec{x}_{cm} \frac{d\ell}{\Lambda^2} \int d\varphi e^{-S_{dipole}} \approx \int d^2 \vec{x}_{cm} \frac{d\ell}{\Lambda^2} \int d\varphi \left( 1 + \frac{2n-2}{1} \sum_{i=1}^{2n-2} K P_i \hat{\varphi} \cdot (\vec{x}_{cm} - \vec{x}_i) + \frac{1}{2} \sum_{i,j=1}^{2n-2} K^2 P_i P_j \hat{\varphi} \cdot \nabla_{x_{cm}} \ln |\vec{x}_i - \vec{x}_{cm}| \hat{\varphi} \cdot \nabla_{x_{cm}} \ln |\vec{x}_j - \vec{x}_{cm}| + \ldots \right)$$

(21)

Since $\varphi$ only appears in $\hat{\varphi}$, the linear term averages to zero. On the other hand, focusing on the second-order term:

$$\int_0^{2\pi} d\varphi \hat{\varphi}_\alpha \hat{\varphi}_\beta = \int_0^{2\pi} d\varphi \delta_{\alpha,\beta} \cos^2 \varphi = \pi \delta_{\alpha,\beta}$$

(22)

and therefore:

$$\rightarrow \int d^2 \vec{x}_{cm} \frac{d\ell}{\Lambda^2} \frac{1}{\Lambda^2} \nabla_{x_{cm}} \ln |\vec{x}_i - \vec{x}_{cm}| \cdot \nabla_{x_{cm}} \ln |\vec{x}_j - \vec{x}_{cm}|$$

(23)

First do integration by parts:

$$\int d^2 \vec{x}_{cm} \nabla \ln |\vec{x}_i - \vec{x}_{cm}| \cdot \nabla_{x_{cm}} \ln |\vec{x}_j - \vec{x}_{cm}| = - \int d^2 \vec{x}_{cm} \ln |\vec{x}_i - \vec{x}_{cm}| \nabla^2 |\vec{x}_j - \vec{x}_{cm}|$$

(24)

And now, for variety we can use an E and M trick:

$$\int d^2 \nabla^2 \ln |\vec{x}| = \int d\delta \cdot \nabla \ln |\vec{x}| = 2\pi$$

(25)

And the answer is:

$$- \int d^2 \vec{x}_{cm} \ln |\vec{x}_i - \vec{x}_{cm}| \delta(\vec{x}_j - \vec{x}_{cm}) = - \ln |\vec{x}_i - \vec{x}_j|$$

(26)

Voilà. This is exactly like the vortex-vortex interaction, and it is going to renormalize the interaction strength.

Our toils are not over yet, though. We now must be very careful. Naively, the correction we calculated is a second order term which scales as $\ln R_{max}$:

$$-K^2 \frac{d\ell}{\Lambda^2} \frac{1}{\Lambda^2} \sum_{i>j} P_i P_j \ln |\vec{x}_j - \vec{x}_j|\Lambda$$

(27)

to a zeroth order term:

$$\frac{d\ell}{\Lambda^2} \frac{1}{2\pi} \int d^2 \vec{x}_{cm} \propto R_{max}^2.$$  

(28)

Nevertheless, both these terms are going to renormalize the $2n - 2$ term:

$$\zeta^{2n-2} Z_{2n-2} \rightarrow \frac{1}{(n-1)!\zeta^{2n-2}} \int \prod_{i=1}^{2n-2} D[\vec{x}_i] e^{- \int \prod_{i=1}^{2n-2} \sum_{i>j} P_i P_j \ln |\vec{x}_i - \vec{x}_j|\Lambda}$$

$$\left(1 + \frac{\zeta^2}{\Lambda^2} \int 2\pi A - K \pi^2 \frac{1}{\Lambda^2} \sum_{i>j} P_i P_j \ln |\vec{x}_j - \vec{x}_j|\Lambda\right) \rightarrow \frac{1}{(n-1)!\zeta^{2n-2}} \int \prod_{i=1}^{2n-2} D[\vec{x}_i] e^{- \int \prod_{i=1}^{2n-2} \sum_{i>j} P_i P_j \ln |\vec{x}_i - \vec{x}_j|\Lambda} + \frac{\zeta^2}{\Lambda^2} \frac{d\ell}{2\pi A}$$

(29)

From which we can see that:

$$K' = K - \frac{\pi^2 K^2}{\Lambda^4} \rightarrow \frac{dK}{d\ell} = -\pi \zeta^2 K^2 \Lambda^{-4}.$$  

(30)

and you can easily convince yourselves that the constant addition to the action does not matter, since the entire partition function becomes:

$$Z(K, \zeta) = e^{\frac{\zeta^2}{\Lambda^2} \int d^2 \pi A} Z(K - \pi \zeta^2 K^2 \Lambda^{-4} d\ell, \zeta(1 + (2 - K/2)d\ell)).$$

(31)

Thus from the consideration of the dipoles, we obtained the decimation’s contribution to the renormalization - $K$ becomes reduced. This contribution is very physical - it is the dielectric response of the vortex-anti-vortex pairs, just like in regular E and M. The dipoles tend to point away from other charges, and thus they reduce the field of any point charges. This adds a flow to the left in the K-\(\zeta\) diagram above, which we analyze next.