Week 5 - Kosterlitz Thouless RG flow and critical properties

I. RG FLOW EQUATIONS

Collecting together the Decimation and rescaling corrections, we find:

\[
\frac{dK}{d\ell} = -2\pi \zeta^2 \Lambda^{-4} K^2
\]

\[
\frac{d\zeta}{d\ell} = \left(2 - \frac{K}{2}\right) \zeta
\]

One can already guess what fixed points would look like - they will have either \(\zeta = 0\) or \(K = 0\). The first is where vortices are paired and at bay, but the second is where they proliferate. By proliferate we mean that the further away you look, The more vortices you see.

The first step in analyzing these equations is to notice that you can chance \(\zeta \rightarrow \zeta' = \alpha \zeta\) without anything happening. In the equation for \(\zeta\) things stay the same, and in the equation for \(K\) we can absorb the change through the cutoff:

\[
\Lambda^2 \rightarrow \Lambda'^2 \rightarrow \alpha \Lambda^2
\]

This is just an indication of the fact that anything that contains the cutoff is non-universal, which means can differ from system to system depending on its microscopic physical properties.

The second step is to say - we are interested only in the vicinity of the phase transition. The \(\zeta = 0\) phase will have to stop where \(K = 4\). Let us define new variables then:

\[
\kappa = \frac{K-4}{2}, \quad y = \zeta \sqrt{8\pi \Lambda^{-2}}
\]

This will reduce the flow equations to:

\[
\frac{d\kappa}{d\ell} = -y^2
\]

\[
\frac{dy}{d\ell} = -\kappa y
\]

It seems that every time that we write an RG equation for two variables, we either get the flow of the epsilon expansion from last term, or this flow here. Luckily, it is not hard to solve. Multiply the top by \(\kappa\) and the bottom by \(y\), and add up the equations:

\[
\kappa \frac{d\kappa}{d\ell} - y \frac{dy}{d\ell} = 0
\]

Integrating this, we get the trajectory of the RG flows:

\[
\kappa^2 - y^2 = \delta
\]

For \(\delta > 0\), \(y\) flows all the way to the \(\kappa\) axis, where \(y\) is zero, and we have the superfluid surviving. In the \(\zeta - K\) plane, this corresponds to a fixed line on the \(K\) axis. This fixed line is stable for \(K > 4\), but it is unstable for \(K < 4\), and results in flows to an insulating vortex proliferated state when \(\zeta > 0\).

The opposite case, \(\delta < 0\), draws trajectories that go towards lower \(K\), or \(\kappa\), first with decreasing \(y\) or \(\zeta\), and then with it increasing. These trajectories begin by the free vortex density dropping, but then the vortex pairs also reduce the stiffness, until that reduction is enough to let vortices get out of control.

The line \(\delta = 0\) is the critical manifold:

\[
\kappa = y
\]

and flows directly to a critical point where the line of fixed points terminate.

So by solving for the RG trajectories, we indeed find the phase diagram - a superfluid fixed line is obtained for this region (shade the right triangle), and an insulator obtains everywhere else. What about critical properties?
II. CORRELATION LENGTHS

A. Superfluid side

How would we define the correlation length? Let’s start with the superfluid phase. In this case, the effective density of free vortices keeps declining. In fact, towards the end of the flow, we have $\zeta \sim e^{-\kappa \ell}$. What are the possible length scales? When we look for a correlation length, we should ask - when is it obvious in which phase we are? Here, it’ll be obvious once the exponential flow obtains.

What scale is that? In the superfluid phase, $\delta > 0$, and its magnitude is a tuning parameter for the K-T transition. For instance, if we begin with $\kappa_0 = y_0$, which is a point on the critical manifold, and deviate from it by setting $K \to K_0 + \Delta K$, or $\kappa + \Delta \kappa$, where $\Delta \kappa \ll \kappa < 1$, then:

$$\kappa^2 - y^2 \approx \kappa_0^2 + 2\kappa_0 \Delta \kappa - \kappa_0^2 = 2\kappa_0 \Delta \kappa = \delta$$  \hspace{1cm} (8)

so:

$$\delta \approx 2\kappa_0 \Delta \kappa$$  \hspace{1cm} (9)

and is a linear tuning parameter for the transition (linear as opposed to, say, proportional to $\Delta K^3$).

Using the RG trajectories:

$$\kappa^2 = y^2 + \delta$$  \hspace{1cm} (10)

and:

$$\frac{dy}{d\ell} = -y\kappa = -y\sqrt{y^2 + \delta}$$  \hspace{1cm} (11)

And from here we can generally solve for the flow:

$$\ell = -\int_{y_0}^{y} \frac{dy}{y\sqrt{\delta + y^2}}$$  \hspace{1cm} (12)

by substituting $y = 1/x$ we get:

$$\ell = \int_{1/y_0}^{1/y} \frac{dx}{\sqrt{\delta x^2 + 1}}$$  \hspace{1cm} (13)

And by substituting $x = \delta^{-0.5} \sinh v$ we obtain:

$$\ell = \frac{1}{\sqrt{\delta}} \int_{\sqrt{\delta}/y_0}^{\sqrt{\delta}/y} dv = \frac{1}{\sqrt{\delta}} \left( \frac{\sqrt{\delta}}{y_0} - \sqrt{\delta} \right)$$  \hspace{1cm} (14)
From which we get the flow:

\[
\frac{\sqrt{\delta}}{y} = \sinh(\sqrt{\delta \ell} + \frac{1}{y_0})
\] (15)

Thus \( \ell_{exp} \), where the flow of \( y \) to zero becomes exponential in \( \ell \), is when the argument of the sinh reaches 1, and:

\[
\xi \sim \frac{1}{\Lambda} e^{1/\sqrt{\delta}}
\] (16)

and we see that \( C \) depends weakly on our choice of \( \epsilon \). This is a correlation length that diverges REALLY quickly. In fact, it is an essential singularity, indicating that the phase transition is extremely feeble and weak. If you have a finite sample, you would hardly feel it.

**B. Insulating side**

What about the other side? On the insulating side, looking at the flow, we can similarly define a length scale associated with when the flow turns from decreasing to increasing \( y \). On the insulating side of the phase diagram \( \delta < 0 \), and \( y^2 = \kappa^2 + |\delta| \). Therefore:

\[
\frac{d\kappa}{d\ell} = -y^2 = -(\kappa^2 + \delta)
\] (17)

This could be integrated:

\[
\int_{\kappa_0}^{0} \frac{d\kappa}{\kappa^2 + |\delta|} = -\ell = -\ln \xi \Lambda
\] (18)

The integral on the left can be done easily using the substitution: \( \kappa = \sqrt{|\delta|} \tan v \):

\[
\Rightarrow \frac{1}{\sqrt{|\delta|}} \int_{0}^{\arctan(\kappa_0/\sqrt{|\delta|})} \frac{dv}{\cos^2 v(1 + \tan^2 v)} = \frac{1}{|\delta|} \arctan(\kappa_0/\sqrt{|\delta|}) \approx \frac{\pi}{2} \frac{1}{\sqrt{|\delta|}} = \ln \xi \Lambda
\] (19)

and we reach the conclusion:

\[
\xi \sim \frac{1}{\Lambda} e^\frac{\pi}{2\sqrt{|\delta|}}
\] (20)

As it turns out, the correlation length which we inferred from the insulating flow is actually quite meaningful physically - it is screening length of the vortex-vortex interaction. Roughly speaking, the interaction of an unpaired vortex plasma is only logarithmic at short scales:

\[
V \sim \ln(|\vec{x}_i - \vec{x}_j|^{-1} + 1/\xi)
\] (21)

and \( \xi \) is roughly the length scale where the strong logarithmic interaction becomes a power law.

**III. UNIVERSAL JUMP OF THE STIFFNESS**

If we were to measure the stiffness of the superfluid film as a function of the bare stiffness \( \chi \), what would we see as we tune \( \chi \) closer to the transition?

The vortices in the film suppress the stiffness, but by how much? This is exactly the question that the RG answers. If we start at a certain point in the parameter space of \( K \) and \( \zeta \), the RG flow diagram takes us down towards the K-axis. Once we get there, we have a description of a vortex-free system, that has a renormalized stiffness - \( \chi = TK/2\pi \).

Now, let us try to draw the resulting stiffness as a function of Temperature as we vary the bare temperature (or stiffness). Once we cross the diagonal critical manifold, we get a flow that goes to zero stiffness! we also get a
proliferation of vortices. So once we cross this line we expect the stiffness to drop to zero. The jump in the stiffness is always

$$\Delta \chi = \frac{2}{\pi} T_{KT}. \quad (22)$$

This is another aspect of the K-T transition: The stiffness jump over the temperature is a universal number, and hence referred to as the “universal jump”. 