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A *geodesic lamination* Λ on a hyperbolic surface Σ is a closed union of disjointly embedded geodesics. One can also define *abstract* 1-dimensional laminations, i.e. spaces which look locally like a product $\mathbb{R} \times K$ where K is some locally compact topological space. We will be interested in the case that K is totally disconnected. We also typically assume that the total space of our lamination Λ is compact. For a geodesic lamination of course, this is implied when Σ is compact.

If Λ is a 1-dimensional (abstract) lamination, we can define an invariant transverse measure μ by taking a sequence of segments in leaves of Λ which are longer and longer, and look at the hitting measure on those segments, scaled by the reciprocal of arclength. For Λ compact, some subsequence converges to an invariant transverse measure. The set of invariant transverse measures supported by a given Λ is the cone on a (Choquet) simplex, whose extremal points are the *ergodic* (projective) transverse measures. Any two ergodic measures μ_1, μ_2 which are not proportional are mutually singular — i.e. there exists a measurable subset of Λ with full measure with respect to μ_1 and zero measure with respect to μ_2. A lamination which supports only one projective class of invariant transverse measure is said to be *uniquely ergodic*.

A lamination is *minimal* if every leaf is dense. Any compact lamination admits a minimal sublamination, by Zorn's lemma. It is interesting to come up with examples of minimal (1-dimensional) laminations which are not uniquely ergodic. In what follows, we give a very simple example.

Define inductively a string of 1's and 2's by the following procedure:

(1) Define $S_1 = 2221$
(2) For any string σ of 1's and 2's define the *complement* of σ, denoted $c(\sigma)$, to be the string obtained from σ by substituting 2 for each 1, and 1 for each 2. E.g. $c(122122) = 211211$
(3) For $n > 1$, define S_n to be the string

$$S_n = S_{n-1}c(S_{n-1})c(S_{n-1}) \cdots c(S_{n-1})$$

where there are $f(n) - 1$ copies of $c(S_{n-1})$, and $f(n)$ is chosen such that $f(1) = 4$, and

$$\prod_n \frac{f(n) - 1}{f(n)} = r > 1/2$$

Then S_i is the initial string of S_{i+1} for each i, and the limit S_∞ has the following properties:

(1) Any finite string which appears in S_∞ appears with density bounded below by some positive constant
(2) The proportion of 2's in S_n is at least r for n odd and at most $1 - r$ for n even

Let Σ be a genus 2 surface, obtained as the union of two 1-holed tori T_1, T_2. Let r be an infinite geodesic ray in Σ obtained from S_∞ as a union of loops in the T_i

representing $(1,1)$ curves, where the first three loops are in T_2, then one loop in T_1, and so on according to the "code" S_∞. r can be pulled tight to a unique geodesic ray, with respect to any hyperbolic structure on Σ.

If r is not embedded, it lifts to an embedded ray in the unit tangent bundle $UT\Sigma$. Let $\bar{r}$ denote the closure of r in $UT\Sigma$. Then $\bar{r}\backslash r$ is an abstract 1-dimensional lamination in $UT\Sigma$, which we denote by Λ. Property (1) implies that Λ is minimal. Property (2) implies that it contains two sequences of segments whose normalized hitting measures converge to μ_1, μ_2 with total mass 1 such that if m_1, m_2 are meridians on T_1, T_2, respectively, then

$$\mu_1(m_1) \geq r, \mu_1(m_2) \leq (1-r),\ \mu_2(m_2) \geq r, \mu_2(m_1) \leq (1-r)$$

In particular, these measures are not proportional, and Λ is minimal but not uniquely ergodic.