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Given a topological space X, we let $C(X)$ denote the set of closed subsets of X. Inclusion of sets defines a natural partial order on $C(X)$.

How much information is contained in this partially ordered set? The answer is simple: all of it.

Lemma 1. *The topological space X can be recovered from the set $C(X)$ together with its partial order.*

Proof. The empty set $\emptyset \in C(X)$ can be identified as the unique element with the property that

$$\emptyset \subseteq K \text{ for all } K \in C(X)$$

So we can identify $C^+(X)$, the set of all nonempty closed subsets of X.

Define a new space Y whose points are the minimal elements (with respect to the partial order) of $C^+(X)$ and whose open sets are those of the form

$$U_K := \{p \in C(X) \mid p \text{ is minimal, and } p \nsubseteq K\}$$

where $K \in C(X)$.

This defines a topology on Y for which it is homeomorphic to X. □

How much of $C(X), \subseteq$ is necessary to recover X? How different are $C(X), C(Y)$ for different spaces X, Y? We make this question very concrete:

Question 2. *Let I denote the closed unit interval, and D the closed unit disk. Is there an injective map*

$$\iota : C(D) \to C(I)$$

which is order preserving?

It turns out that this question is very easy: there are many such maps.

Example 3. Let $\phi : I \to D$ be a continuous, surjective map. Then the map

$$K \to \phi^{-1}(K)$$

defines an injective order-preserving map from $C(D) \to C(I)$.

Note that we have no control over the relationship between the topology of $\phi^{-1}(K), \phi^{-1}(L) \subset I$ for homeomorphic K, L. This motivates the following definition and question.

Definition 4. Two subsets $K, L \in C(X)$ are *abstractly homeomorphic* if they are homeomorphic as topological spaces. They are *ambiently homeomorphic* if there is a homeomorphism $h : X \to X$ such that $h(K) = L$.

Question 5. *Is there a continuous, surjective map $\phi : I \to D$ such that whenever $K, L \in C(D)$ are ambiently homeomorphic, $\phi^{-1}(K), \phi^{-1}(L)$ are abstractly homeomorphic?*

One could also modify this question by replacing the first occurence of "ambiently" by "abstractly".

Note that every homeomorphism of I must permute the two endpoints, which we denote I^+ and I^-; it follows that one cannot hope for such a map for which $\phi^{-1}(K)$ and $\phi^{-1}(L)$ are ambiently homeomorphic whenever K, L are. For, if $p^\pm = \phi(I^\pm)$ then for any $q \in D$ not equal to p^+ or p^-, the preimage $\phi^{-1}(q)$ contains no endpoint, and is therefore not ambiently homeomorphic to $\phi^{-1}(p^+)$.

As a first attempt, we restrict attention to points and point preimages.

Question 6. *Is there a continuous, surjective map $\phi : I \to D$ such that any two point preimages are abstractly homeomorphic?*

Any two Cantor sets are abstractly homeomorphic, and any two Cantor sets in I are ambiently homeomorphic if they share the same number of endpoints. So one could be bold and ask

Question 7. *Is there a continuous, surjective map $\phi : I \to D$ such that every point preimage is a Cantor set?*

We show that the answer to this question is *yes*.

Theorem 8. *There is a continuous, surjective map $\phi : I \to D$ such that every point preimage is a Cantor set.*

Proof. We first construct a surjective map $f : I \to I$ such that every point preimage is a Cantor set.

We define f as the limit (in the compact-open topology) of a sequence of maps f_n defined recursively as follows.

(1) f_0 is defined by

$$f_0(x) = \begin{cases} 2x & \text{if } x < 1/2 \\ 2 - 2x & \text{if } x \geq 1/2 \end{cases}$$

(2) f_i is piecewise linear, and is linear on each segment

$$J^i_p := [p/2 \cdot 6^i, (p+1)/2 \cdot 6^i] \, , \; 0 \leq p < 2 \cdot 6^i$$

Given f_i, we define f_{i+1} on J^i_p as follows. Let $J(t)$ parameterize J^i_p, where $J(0)$ is the initial point, and $J(1)$ is the terminal point. We need to specify the value of f_{i+1} on each point of the form $J(q/6)$.

$$f_{i+1}(x) = \begin{cases} f_i(J(0)) & \text{if } x = J(0) \\ f_i(J(1/2)) & \text{if } x = J(1/6) \\ f_i(J(0)) & \text{if } x = J(2/6) \\ f_i(J(1/2)) & \text{if } x = J(3/6) \\ f_i(J(1)) & \text{if } x = J(4/6) \\ f_i(J(1/2)) & \text{if } x = J(5/6) \\ f_i(J(1)) & \text{if } x = J(1) \end{cases}$$

In words: each segment of the graph of f_i over J^i_p is split into two subsegments. Each such subsegment is replaced by an "N" if it slopes up, or the mirror image of "N" if it slopes down. See figure 1.

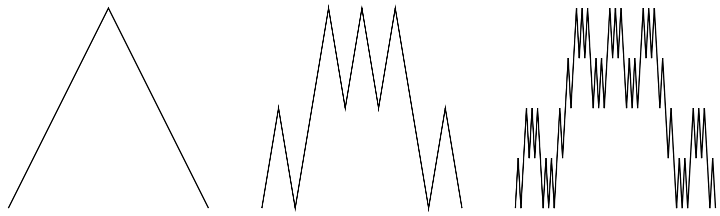

FIGURE 1. The graphs of f_0, f_1, f_2

Defining f to be the limit, we see that f has the desired properties.

Suppose $g : I \to D$ is a surjective, continuous map such that each point in D has only finitely many preimages. Such maps are easy to construct. For example, let M be a hyperbolic 3-manifold which fibers over S^1 with fiber Σ. Then there is a continuous surjective map from the circle at infinity of $\widetilde{\Sigma}$ to the sphere at infinity of $\widetilde{M}$, by Cannon-Thurston. Restricting attention to a subinterval of S^1 bounded by a leaf of the stable lamination, the image can be taken to be a disk bounded by a quasicircle. This is the desired map g.

Now, define $h : I \to D$ by $h = gf$. Each point preimage of h is a finite union of Cantor sets in I. A finite union of Cantor sets is closed and perfect, and has no interior, and is therefore itself a Cantor set. $\square$