Physics 106b:EM Assignment 2 — Solutions

1. (a) Since system has translational invariant symmetry in z direction, V only depends on x,y

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{1}$$

seperation variables and using boundary conditions

$$V(x,y) = \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi y}{a}) \sinh(\frac{n\pi x}{a})$$

when $x = a$,

$$V(x,y) = V_0(y) = \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi y}{a}) \sinh(n\pi)$$

$$C_n = \frac{2}{a \sinh(n\pi \frac{b}{a})} \int_0^a V_0(y) \sin(\frac{n\pi y}{a})\, dy \tag{2}$$

so we get

$$V(x,y) = \sum_{n=1}^{\infty} \frac{2\sinh(\frac{n\pi x}{a})\sin(\frac{n\pi y}{a})}{a \sinh(n\pi \frac{b}{a})} \int_0^a V_0(y) \sin(\frac{n\pi y}{a})\, dy \tag{3}$$

(b)

$$V(x,y) = \sum_{n=1}^{\infty} \frac{2\sinh(\frac{n\pi x}{a})\sin(\frac{n\pi y}{a})}{a \sinh(n\pi \frac{b}{a})} (\frac{V_0 a}{n\pi})(1-(-1)^n) \tag{4}$$

$$V(x,y) = \sum_{m=1}^{\infty} (\frac{4V_0}{(2m+1)\pi}) \frac{\sinh(\frac{(2m+1)\pi x}{a})\sin(\frac{(2m+1)\pi y}{a})}{\sinh((2m+1)\pi \frac{b}{a})} \tag{5}$$

2.

$$E = \frac{\partial V(x,y)}{\partial x}|_{x=0} = \sum_{m=1}^{\infty} (\frac{4V_0}{(2m+1)\pi}) \frac{\sin(\frac{(2m+1)\pi y}{a})}{\sinh((2m+1)\pi \frac{b}{a})} (\frac{(2m+1)\pi}{a}) \tag{6}$$

so that

$$E = \sum_{m=1}^{\infty} (\frac{4V_0}{a}) \frac{\sin(\frac{(2m+1)\pi y}{a})}{\sinh((2m+1)\pi \frac{b}{a})} \tag{7}$$

$$Q = \epsilon_0 \int_0^a E\, dy$$

$$Q = \sum_{m=1}^{\infty} \frac{8V_0\epsilon_0}{\pi} \frac{1}{2m+1} \frac{1}{\sinh(2m+1)\pi} \tag{8}$$

3. (a) Due to translational symmetry in y direction $V = V(x,z)$,given the period in x direction, we can write

$$V(x,z) = (A \sin kx + B \cos kx) e^{-k|z|} \tag{9}$$

from Gaus Theorem

$$\frac{\partial V}{\partial z}|_{z=0^+} = \frac{\sigma_0}{2\epsilon_0}$$

we get

$$V(x,z) = -\frac{\sigma_0 \sin kx}{2\epsilon_0 k} e^{-k|z|} \tag{10}$$

when$z \gg \frac{1}{k}$, v is small

(b)

$$E_x = \frac{\partial V}{\partial x} = -\frac{\sigma_0 \cos kx}{2\epsilon_0} e^{-k|z|}$$

$$E_y = 0$$

$$E_z = \frac{\partial V}{\partial z}$$

$$E_z = \begin{cases} \frac{\sigma_0 \sin kx}{2\epsilon_0} e^{-k|z|} & \text{if } z > 0, \\ -\frac{\sigma_0 \sin kx}{2\epsilon_0} e^{-k|z|} & \text{otherwise.} \end{cases}$$

(c)

$$E^2 = {E_x}^2 + {E_z}^2 = (\frac{\sigma_0}{2\epsilon_0})^2 e^{-2k|z|} \tag{11}$$

$$U = \int_{-\infty}^{\infty} \frac{\epsilon_0 E^2}{2}\, dz = \frac{{\sigma_0}^2}{8\epsilon_0 k} \tag{12}$$

4. Due to symmetry of the system,the mid plane must have potential $\frac{V_0}{2}$, consider the bottom half

$$V(x,y) = \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi x}{a}) \sinh[\frac{(y+a/2)n\pi}{a}] \tag{13}$$

when$y = 0$,

$$V = \frac{V_0}{2} = \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi x}{a}) \sinh(\frac{n\pi}{2})$$

then we can get

$$C_n = \frac{V_0}{2\sinh(\frac{n\pi}{2})}(\frac{2}{a}) \int_0^a \sin(\frac{n\pi x}{a})\, dx = \frac{V_0(1-(-1)^n)}{n\pi \sinh(\frac{n\pi}{2})} \tag{14}$$

$$\longrightarrow V(x.y) = \sum_{m=1}^{\infty} \frac{2V_0}{(2m+1)\pi} \frac{\sinh[\frac{(2m+1)(y+a/2)\pi}{a}] \sin[\frac{(2m+1)\pi x}{a}]}{\sinh[\frac{(2m+1)\pi}{2}]}$$

Due to the symmetry

$$V(x,y) + V(x,-y) = V_0$$

so for the upper half

$$V(x.y) = V_0 + \sum_{m=1}^{\infty} \frac{2V_0}{(2m+1)\pi} \frac{\sinh[\frac{(2m+1)(y-a/2)\pi}{a}] \sin[\frac{(2m+1)\pi x}{a}]}{\sinh[\frac{(2m+1)\pi}{2}]} \tag{15}$$