Physics 106b – Problem Set 7 – Due Mar 6, 2009
Solutions

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Problem 1

(a) Using the recursion relation,

$$(2l+1)xP_l(x) = (l+1)P_{l+1}(x) + lP_{l-1}(x)$$

We can get

$$P_{l+1}(0) = -\frac{l}{l+1}P_{l-1}(0) = \frac{l}{l+1}\frac{l-2}{l-1}P_{l-3}(0) = \cdots = (-1)^{\frac{l+1}{2}}\frac{l!!}{(l+1)!!}P_0(0) = (-1)^{\frac{l+1}{2}}\frac{l!!}{(l+1)!!}$$

for odd l

(b) From the lecture, we have

$$\int_0^1 P_l(x)dx = -\frac{P_{l+1}(0)}{l} = -\frac{1}{l}(-1)^{\frac{l+1}{2}}\frac{l!!}{(l+1)!!} = (-1)^{\frac{l-1}{2}}\frac{(l-2)!!}{(l+1)!!}$$

Problem 2

This problem is exactly the same as Example 3.6 in the textbook. Therefore the solution is

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

where A_l is given by Eq. (3.69).

$$\begin{aligned}
A_l &= \frac{2l+1}{2R^l}\int_0^{\pi} V_0(\theta)P_l(\cos\theta)\sin\theta d\theta \\
&= \frac{2l+1}{2R^l}\int_0^{\pi/2} V_0P_l(\cos\theta)\sin\theta d\theta - \int_{\pi/2}^{\pi} V_0P_l(\cos\theta)\sin\theta d\theta \\
&= \frac{2l+1}{2R^l}\int_0^{1} V_0P_l(x)dx - \int_{-1}^{0} V_0P_l(x)dx \\
&= \begin{cases} \frac{2l+1}{R^l}\int_0^1 V_0P_l(x)dx & \text{if } l \text{ is odd} \\ 0 & \text{if } l \text{ is even} \end{cases} \\
&= \begin{cases} \frac{2l+1}{R^l}V_0(-1)^{\frac{l-1}{2}}\frac{(l-2)!!}{(l+1)!!} & \text{if } l \text{ is odd} \\ 0 & \text{if } l \text{ is even} \end{cases}
\end{aligned}$$

Problem 3

(a) The first part is trivial.

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{d/2 - z} - \frac{1}{4\pi\epsilon_0}\frac{q}{d/2 + z}$$

(b)

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{|\mathbf{r} - d/2\hat{\mathbf{z}}|} - \frac{1}{4\pi\epsilon_0}\frac{q}{|\mathbf{r} + d/2\hat{\mathbf{z}}|}$$

Using Eq. (3.94), we have

$$\begin{aligned}
\frac{1}{|\mathbf{r} - d/2\hat{\mathbf{z}}|} &= \frac{1}{d/2}\sum_{n=0}^{\infty}\left(\frac{r}{d/2}\right)^n P_n(\cos\theta) \\
\frac{1}{|\mathbf{r} + d/2\hat{\mathbf{z}}|} &= \frac{1}{d/2}\sum_{n=0}^{\infty}\left(\frac{r}{d/2}\right)^n P_n(\cos(\pi - \theta)) \\
&= \frac{1}{d/2}\sum_{n=0}^{\infty}\left(\frac{r}{d/2}\right)^n P_n(-\cos\theta)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow V &= \frac{1}{4\pi\epsilon_0}\frac{q}{|\mathbf{r} - d/2\hat{\mathbf{z}}|} - \frac{1}{4\pi\epsilon_0}\frac{q}{|\mathbf{r} + d/2\hat{\mathbf{z}}|} \\
&= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{d/2}\sum_{n=0}^{\infty}\left(\frac{r}{d/2}\right)^n (P_n(\cos\theta) - P_n(-\cos\theta))\right] \\
&= \frac{q}{4\pi\epsilon_0}\left[\frac{4}{d}\sum_{n=0}^{\infty}\left(\frac{r}{d/2}\right)^{2n+1} P_{2n+1}(\cos\theta)\right]
\end{aligned}$$

Problem 4

From Eq. (3.65)

$$\begin{aligned} V(r,\theta) &= \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta) \\ \Rightarrow V(a,\theta) &= \sum_{l=0}^{\infty}\left(A_l a^l + \frac{B_l}{a^{l+1}}\right)P_l(\cos\theta) = V_a P_3(\cos\theta) \\ V(b,\theta) &= \sum_{l=0}^{\infty}\left(A_l b^l + \frac{B_l}{b^{l+1}}\right)P_l(\cos\theta) = V_b P_5(\cos\theta) \\ &\Rightarrow \begin{cases} A_3a^3 + \frac{B_3}{a^4} = V_a \\ A_l a^l + \frac{B_l}{a^{l+1}} = 0 \quad \text{for } l \neq 3 \\ A_5 b^5 + \frac{B_5}{b^6} = V_b \\ A_l b^l + \frac{B_l}{b^{l+1}} = 0 \quad \text{for } l \neq 5 \end{cases} \\ &\Rightarrow \begin{cases} A_3a^3 + \frac{B_3}{a^4} = V_a \\ A_5 a^5 + \frac{B_5}{a^6} = 0 \\ A_5 b^5 + \frac{B_5}{b^6} = V_b \\ A_3 b^3 + \frac{B_3}{b^4} = 0 \end{cases} \\ &\Rightarrow \begin{cases} A_3 = \frac{V_a/b^4}{a^3/b^4 - b^3/a^4} \\ A_5 = \frac{V_b/a^6}{b^5/a^6 - a^5/b^6} \\ B_3 = \frac{V_a b^3}{b^3/a^4 - a^3/b^4} \\ B_5 = \frac{V_b a^5}{a^5/b^6 - b^5/a^6} \\ A_l = B_l = 0 \quad \text{for } l \neq 3 \text{ } or \text{ } 5 \end{cases} \\ &\therefore \quad V(r,\theta) = \left(A_3 r^3 + \frac{B_3}{r^4}\right)P_3(\cos\theta) + \left(A_5 r^5 + \frac{B_5}{r^6}\right)P_5(\cos\theta) \end{aligned}$$

Problem 5

First, we express the dipole in $x-y-z$ basis. $\mathbf{p} = p\cos\theta\hat{\mathbf{z}} + p\sin\theta\hat{\mathbf{x}}$. Secondly, we use the method of images to do this problem. If we put another dipole $\mathbf{p}' = p\cos\theta\hat{\mathbf{z}} - p\sin\theta\hat{\mathbf{x}}$ at $(0,0,-z)$, then the potential on the conducting plane is held to be zero. By the uniqueness theorem, this configuration produce exactly the same potential as the original one. Using Eq. (3.104), dipole $\mathbf{p}'$ causes the electric field at $(0, 0, z)$

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\frac{1}{(2z)^3}\left[(3\mathbf{p}'\cdot\hat{\mathbf{z}})\hat{\mathbf{z}} - \mathbf{p}'\right] \\
&= \frac{1}{4\pi\epsilon_0}\frac{1}{(2z)^3}\left[(3p'\cos\theta)\hat{\mathbf{z}} - (p\cos\theta\hat{\mathbf{z}} - p\sin\theta\hat{\mathbf{x}})\right] \\
&= \frac{p}{32\pi\epsilon_0}\frac{1}{z^3}(2\cos\theta\hat{\mathbf{z}} + \sin\theta\hat{\mathbf{x}})
\end{aligned}$$

From Eq. (4.4), we have

$$\begin{aligned}
\mathbf{N} &= \mathbf{p}\times\mathbf{E} = (p\cos\theta\hat{\mathbf{z}} + p\sin\theta\hat{\mathbf{x}})\times\left[\frac{p}{32\pi\epsilon_0}\frac{1}{z^3}(2\cos\theta\hat{\mathbf{z}} + \sin\theta\hat{\mathbf{x}})\right] \\
&= -\frac{p^2\sin\theta\cos\theta\hat{\mathbf{y}}}{32\pi\epsilon_0 z^3}
\end{aligned}$$

We should consider two different cases if the dipole is free to rotate, $0 < \theta_0 < \pi/2$ and $\pi/2 < \theta_0 < \pi$. For the first case, the torque is on the $-y$ direction initially, so the dipole will rotate counterclockwise, and finally it will rest at $\theta = 0$. By the same analysis, for $\pi/2 < \theta_0 < \pi$ case, it will rest at $\theta = \pi$ finally.

Problem 6

Use Eq. (3.104), we have

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\mathbf{p}_2\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}_2\right]$$

Use Eq. (4.7),

$$U = -\mathbf{p}_1\cdot\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[\mathbf{p}_1\cdot\mathbf{p}_2 - 3(\mathbf{p}_1\cdot\hat{\mathbf{r}})(\mathbf{p}_2\cdot\hat{\mathbf{r}})\right]$$