

Physics 106c – Problem Set 1 – Due Apr. 10, 2009
Solutions

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Problem 1

If we assume $\phi(z=0) = 0$, then on the symmetry axis we have

$$\mathbf{B} = \nabla\phi \Rightarrow \phi = \int \mathbf{B} \cdot d\mathbf{l} = \int_0^z \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z^2)^{3/2}} dz = \frac{\mu_0 I}{2} \frac{z}{\sqrt{R^2 + z^2}}$$

When $z \gg R$

$$\begin{aligned} \phi(\text{on the symmetry axis}) &= \frac{\mu_0 I}{2} \frac{z}{\sqrt{R^2 + z^2}} = \frac{\mu_0 I}{2} \left(1 - \frac{1}{2} \frac{R^2}{z^2} + \frac{3}{8} \frac{R^4}{z^4} - \frac{5}{16} \frac{R^6}{z^6} + \dots \right) \\ \Rightarrow \phi(\mathbf{r}) &= \frac{\mu_0 I}{2} \left(1 - \frac{1}{2} \frac{R^2}{r^2} P_1(\cos\theta) + \frac{3}{8} \frac{R^4}{r^4} P_3(\cos\theta) - \frac{5}{16} \frac{R^6}{r^6} P_5(\cos\theta) + \dots \right) \\ \therefore \mathbf{B} = \nabla\phi &= \frac{\partial\phi}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial\phi}{\partial\theta} \hat{\theta} \\ &= \frac{\mu_0 I}{2} \left(\frac{R^2}{r^3} P_1(\cos\theta) - \frac{3}{2} \frac{R^4}{r^5} P_3(\cos\theta) + \frac{15}{8} \frac{R^6}{r^7} P_5(\cos\theta) + \dots \right) \hat{\mathbf{r}} \\ &\quad + \frac{\mu_0 I}{2} \left(-\frac{1}{2} \frac{R^2}{r^3} \frac{dP_1(\cos\theta)}{d\theta} + \frac{3}{8} \frac{R^4}{r^5} \frac{dP_3(\cos\theta)}{d\theta} - \frac{5}{16} \frac{R^6}{r^7} \frac{dP_5(\cos\theta)}{d\theta} + \dots \right) \hat{\theta} \end{aligned}$$

Therefore, at very large distance, the first order term of the magnetic field is

$$\mathbf{B} = \frac{\mu_0 I}{2} \left(\frac{R^2}{r^3} P_1(\cos\theta) \hat{\mathbf{r}} - \frac{1}{2} \frac{R^2}{r^3} \frac{dP_1(\cos\theta)}{d\theta} \hat{\theta} \right) = \frac{\mu_0 I \pi R^2}{4\pi r^3} \left(2 \cos\theta \hat{\mathbf{r}} + \sin\theta \hat{\theta} \right)$$

,which is exactly the magnetic field of a simple dipole with dipole moment $m = I\pi R^2$.

Problem 2

$$\begin{aligned}
 \mathbf{A} &= \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\tau' \\
 \Rightarrow \nabla \cdot \mathbf{A} &= \frac{\mu_0}{4\pi} \int \nabla \cdot \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) d\tau' \\
 &= \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \cdot \nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) d\tau' \\
 &= -\frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \cdot \nabla' \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) d\tau' \\
 &= -\frac{\mu_0}{4\pi} \int \left(\nabla' \cdot \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) - \frac{\nabla' \cdot \mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) d\tau' \\
 &= -\frac{\mu_0}{4\pi} \left(\oint \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \cdot d\mathbf{a} - \int \frac{\nabla' \cdot \mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\tau' \right)
 \end{aligned}$$

Because the current distribution is localized, the surface integral of the first term is zero. Furthermore, the second term is also zero because the magnetostatics condition. ($\nabla \cdot \mathbf{J} = 0$)

Problem 3

Use Eq. (5.38), we have

$$\begin{aligned}
 B(z) &= \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z^2)^{3/2}} \\
 dB &= \frac{\mu_0 dI}{2} \frac{R^2}{(R^2 + z^2)^{3/2}} \\
 &= \frac{\mu_0 n I dz}{2} \frac{R^2}{(R^2 + z^2)^{3/2}} \\
 \therefore B &= \int \frac{\mu_0 n I dz}{2} \frac{R^2}{(R^2 + z^2)^{3/2}} \\
 &= \frac{\mu_0 I n}{2} (\cos \theta_1 - \cos \theta_2)
 \end{aligned}$$

(Ps. See the figure 5.25)

Problem 4

(a)

$$\begin{aligned}
 \mathbf{B}_{ave} &= \frac{1}{\frac{4}{3}\pi R^3} \int \mathbf{B} d\tau = \frac{1}{\frac{4}{3}\pi R^3} \int (\nabla \times \mathbf{A}) d\tau \\
 &= -\frac{1}{\frac{4}{3}\pi R^3} \int_s \mathbf{A} \times d\mathbf{a} = -\frac{1}{\frac{4}{3}\pi R^3} \int_s \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') d\tau'}{|\mathbf{r} - \mathbf{r}'|} \times d\mathbf{a}
 \end{aligned}$$

We have to do the surface integral first. By symmetry, we can easily argue that $\int_s \frac{1}{|\mathbf{r}-\mathbf{r}'|} d\mathbf{a}$ is proportional to (or in the same direction of) \mathbf{r}' , which means

$$\int_s \frac{1}{|\mathbf{r}-\mathbf{r}'|} d\mathbf{a} = \text{const} \cdot \mathbf{r}'$$

Therefore,

$$\begin{aligned} \int_s \frac{1}{|\mathbf{r}-\mathbf{r}'|} d\mathbf{a} &= \int_s \frac{1}{|\mathbf{r}-\mathbf{r}'|} (\hat{\mathbf{r}}' \cdot d\mathbf{a}) \hat{\mathbf{r}}' \\ &= \int_0^{2\pi} d\phi \int_0^\pi d\theta \frac{R^2 \sin\theta \cos\theta}{\sqrt{r'^2 + R^2 - 2Rr' \cos\theta}} \hat{\mathbf{r}}' \\ &= 2\pi R^2 \frac{(r' + R)(r'^2 - r'R + R^2) - (|r' - R|)(r'^2 + r'R + R^2)}{3r'^2 R^2} \hat{\mathbf{r}}' \\ &= \begin{cases} \frac{4\pi}{3} \mathbf{r}' & \text{if } r' < R \\ \frac{4\pi}{3} \frac{R^3}{r'^3} \mathbf{r}' & \text{if } r' > R \end{cases} \end{aligned}$$

For currents are all within the sphere, $r' < R$

$$\begin{aligned} \Rightarrow \mathbf{B}_{ave} &= -\frac{1}{\frac{4}{3}\pi R^3} \int_s \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') d\tau'}{|\mathbf{r}-\mathbf{r}'|} \times d\mathbf{a} \\ &= -\frac{1}{\frac{4}{3}\pi R^3} \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \times \frac{4\pi}{3} \mathbf{r}' d\tau' \\ &= \frac{\mu_0}{4\pi} \frac{2\mathbf{m}}{R^3} \end{aligned}$$

(b) For currents are all outside the sphere ($r' > R$), we have

$$\begin{aligned} \mathbf{B}_{ave} &= -\frac{1}{\frac{4}{3}\pi R^3} \int_s \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') d\tau'}{|\mathbf{r}-\mathbf{r}'|} \times d\mathbf{a} \\ &= -\frac{1}{\frac{4}{3}\pi R^3} \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \times \frac{4\pi}{3} \frac{R^3}{r'^3} \mathbf{r}' d\tau' \\ &= -\frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \times \left(\frac{\hat{\mathbf{r}}'}{r'^2} \right) d\tau' \\ &= \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \times \nabla' \left(\frac{1}{r'} \right) d\tau' \end{aligned}$$

The field which the steady currents produce is

$$\begin{aligned} \mathbf{B}(\mathbf{r}) &= \nabla \times \mathbf{A} = \frac{\mu_0}{4\pi} \int \nabla \times \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \right) d\tau' \\ &= -\frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \times \nabla \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) d\tau' \\ &= \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \times \nabla' \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) d\tau' \\ \Rightarrow \mathbf{B}(\mathbf{0}) &= \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \times \nabla' \left(\frac{1}{r'} \right) d\tau' \\ &= \mathbf{B}_{ave} \end{aligned}$$

Problem 5

(a)

$$\begin{aligned}\mathbf{J}_b &= \nabla \times \mathbf{M} = -k\hat{\phi} \\ \mathbf{K}_b &= \mathbf{M} \times \hat{\mathbf{n}} = kR\hat{\phi}\end{aligned}$$

Then we can use Ampere's law to find the magnetic field. Using the same argument and Ampere loop as example 5.9, we can easily get

$$\mathbf{B} = \begin{cases} \mu_0 k s \hat{\mathbf{z}} & s \leq R \\ 0 & s > R \end{cases}$$

(b) Because there is no free current anywhere, we can easily deduce $\mathbf{H} = 0$ everywhere. Using $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$, we can get

$$\mathbf{B} = \begin{cases} \mu_0 k s \hat{\mathbf{z}} & s \leq R \\ 0 & s > R \end{cases}$$

the same as (a)

Problem 6

- (a) From Eq. (6.16), we know that the field inside the sphere with magnetization $-\mathbf{M}$ is $-\frac{2}{3}\mu_0\mathbf{M}$. Therefore, by superposition principle, the field at the center of the cavity is $(\mathbf{B}_0 - \frac{2}{3}\mu_0\mathbf{M})$. Furthermore, $\mathbf{H} = \frac{\mathbf{B}}{\mu_0} = \mathbf{H}_0 + \frac{1}{3}\mathbf{M}$
- (b) Using Eq. (6.20), it is easy to get that the field inside a long needle-shaped material with magnetization $-\mathbf{M}$ is $-\mu_0\mathbf{M}$. Therefore, the field at the center of the cavity is $(\mathbf{B}_0 - \mu_0\mathbf{M})$, and $\mathbf{H} = \frac{\mathbf{B}}{\mu_0} = \mathbf{H}_0$
- (c) The same method as above, the field at the center of the cavity is \mathbf{B}_0 , and $\mathbf{H} = \frac{\mathbf{B}}{\mu_0} = \mathbf{H}_0 + \mathbf{M}$