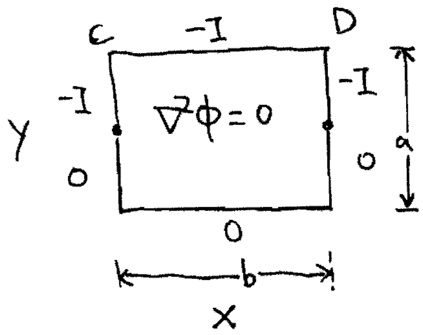
Ph106C Set 2

1. $\vec{K} = \hat{z} \times \nabla\phi \Rightarrow K_x = -\frac{\partial \phi}{\partial y}, \; K_y = \frac{\partial \phi}{\partial x}$

$\nabla \times \vec{E} = 0 \Rightarrow \nabla^2 \phi = 0$

Now the boundary value problem looks like



This problem has been solved last term

For $\frac{a}{2} < y < a$

$$\phi(x,y) = -I + \frac{2I}{\pi} \sum_{n\,odd} \frac{\sin(\frac{n\pi x}{b}) \sinh(\frac{n\pi(a-y)}{b})}{n \sinh(\frac{n\pi a}{2b})},$$

For $0 < y < \frac{a}{2}$, $\phi(x,y) = -\frac{2I}{\pi} \sum_{n\,odd} \frac{\sinh(\frac{n\pi y}{b}) \sin(\frac{n\pi x}{b})}{n \sinh(\frac{n\pi a}{2b})}$

$$\Delta V_{cd} = \int E\, dl = \frac{1}{\sigma} \int_0^b -\frac{\partial \phi}{\partial y}(x, y=a)\, dx$$

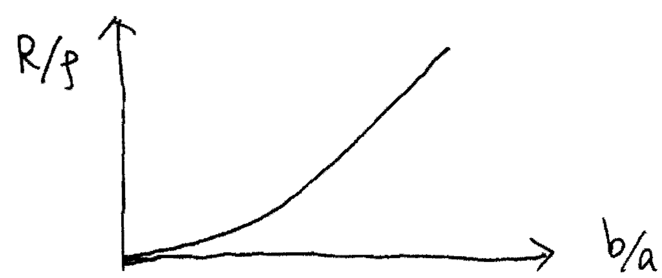
$$= \rho \frac{4I}{\pi} \sum_{n\,odd} \frac{1}{n \sinh(\frac{n\pi a}{2b})}, \Rightarrow R = \frac{4\rho}{\pi} \sum_{n\,odd} \frac{1}{n \sinh(\frac{n\pi a}{2b})}$$

when $b/a \to \infty$

$$R \to \rho \cdot \frac{4}{\pi} \cdot \frac{2b}{\pi a} \sum_{n\,odd} \frac{1}{n^2}$$

$$= \rho \frac{8b}{\pi^2 a} \cdot \frac{\pi^2}{8} = \rho\, b/a$$

R/ρ v.s b/a is like



Solution for Homework #2

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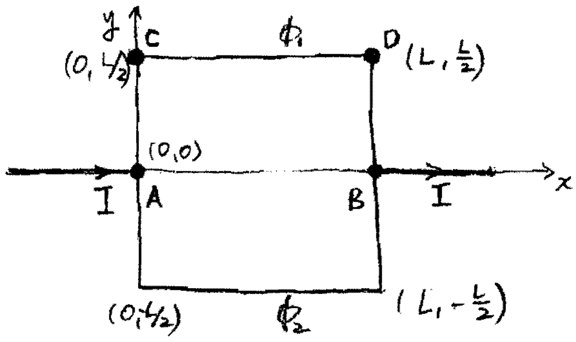


Since currents don't go out vertically at each edge,

$K_x(0, y) = K_x(L, y) = 0$ for $y \neq 0$, $y \in [L/2, -L/2]$

$K_y(x, L/2) = K_y(x, -L/2) = 0$ for $x \in [0, L]$

From $\vec{K} = \hat{z} \times \vec{\nabla}\phi$, $K_x = -\frac{\partial \phi}{\partial y}$, $K_y = \frac{\partial \phi}{\partial x}$.

Therefore, $\left.\frac{\partial \phi}{\partial y}\right|_{x=0,L} = 0$ for $y \in [-L/2, 0) \cup (0, L/2]$ and ϕ is constant at upper edge (ϕ_1) and lower edge (ϕ_2). So if we integrate K_x over $y \in [L/2, -L/2]$, we have I and get condition for ϕ_1 and ϕ_2.

$$I = \int_{-L/2}^{L/2} K_x \, dy = -\int_{-L/2}^{L/2} \frac{\partial \phi}{\partial y} dy = -(\phi_1 - \phi_2) = \phi_2 - \phi_1$$

We don't lose generality though we choose $\phi_2 = I$, $\phi_1 = 0$

Since $\vec{\nabla} \times \vec{E} = 0$ in this case (no time-varying $\vec{B}$ field), and $E_x = \rho_{xx} K_x$, $E_y = \rho_{yy} K_y$

$$\Rightarrow \frac{\partial K_y}{\partial x} - \frac{\rho_{xx}}{\rho_{yy}} \frac{\partial K_x}{\partial y} = 0 \Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\rho_{xx}}{\rho_{yy}} \frac{\partial^2 \phi}{\partial y^2} = 0$$

By using seperation of variables ($\phi = X(x)Y(y)$), $\frac{X''}{X} = -\frac{\rho_{xx}}{\rho_{yy}} \frac{Y''}{Y} = -k^2$

Our boundary conditions give $X(x)Y(L/2) = 0$, $X(x)Y(-L/2) = I$, — ①

$X(0)Y'(y) = X(L)Y'(y) = 0$ $y \in [-L/2, 0) \cup (0, L/2]$ — ②

Then

$X(x) = A\sin kx + B\cos kx$, $Y(y) = Ce^{k\eta y} + De^{-k\eta y}$ where $\eta = \sqrt{\frac{\rho_{yy}}{\rho_{xx}}}$

② $\Rightarrow$ $X(x) \sim \sin(\frac{n\pi}{L}x)$

① $\Rightarrow$ $C_n e^{n\pi\eta/2} + D_n e^{-n\pi\eta/2} = 0$ $C_n \sim e^{-n\pi\eta/2}$, $D_n \sim -e^{n\pi\eta/2}$

$\Rightarrow$ $Y(y) \sim \sinh(\frac{n\pi\eta}{L}(y - \frac{L}{2}))$

$$\Rightarrow \phi(x, y) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{L}x) \sinh(\frac{n\pi\eta}{L}(y - L/2))$$

By symmetry and $\Phi_2 = I$ and $\Phi_1 = 0$, we have $\phi(x,0) = \frac{I}{2}$

$$\Rightarrow \frac{I}{2} = -\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) \sinh\left(\frac{n\pi\eta}{2}\right) \Rightarrow A_n = \frac{-I}{n\pi \sinh(n\pi\eta/2)}(1-(-1)^n),$$

where we used $\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \frac{L}{2}\delta_{mn}$ at the last step.

Hence

$$\phi(x,y) = \sum_{n=1}^{\infty} \frac{I}{n\pi}(1-(-1)^n) \frac{\sinh\left(\frac{n\pi\eta}{L}(L/2 - y)\right)}{\sinh(n\pi\eta/2)} \sin\left(\frac{n\pi x}{L}\right)$$

Thus,

$$V_{CD} = \int_0^L E_x(x, L/2)\, dx = -\rho_{xx}\int_0^L \left.\frac{\partial\phi}{\partial y}\right|_{y=L/2} dx = \rho_{xx}\frac{\eta I}{L}\sum_{n=1}^{\infty}(1-(-1)^n)\frac{1}{\sinh(n\pi\eta/2)}\int_0^L \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= I\sqrt{\rho_{xx}\rho_{yy}}\sum_{n=1}^{\infty}\frac{(1-(-1)^n)^2}{\sinh(n\pi\eta/2)} \cdot \frac{1}{n\pi}$$

$$R = \frac{V_{CD}}{I} = (\rho_{xx}\rho_{yy})^{1/2}\sum_{n=1}^{\infty}\frac{1}{n\pi}(1-(-1)^n)^2 \frac{1}{\sinh(n\pi\sqrt{\rho_{yy}/\rho_{xx}}/2)}$$

(i) $\rho_{xx} = 1 \quad \rho_{yy} = 0.2$

$\Rightarrow R_1 = 0.802658$ $\quad \Rightarrow \frac{R_1}{R_2} = 23.6037 \quad \frac{\rho_{xx}^{(1)}}{\rho_{xx}^{(2)}} = 5 \quad \frac{\rho_{yy}^{(1)}}{\rho_{yy}^{(2)}} = 0.2$

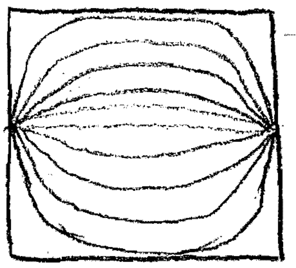
(ii) $\rho_{xx} = 0.2 \quad \rho_{yy} = 1$

$\Rightarrow R_2 = 0.0340057$

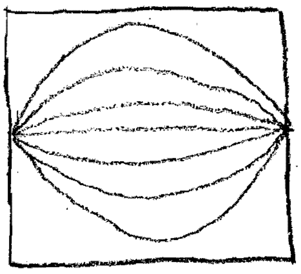
So there is a difference between R_1/R_2 & $\rho_{xx}^{(1)}/\rho_{xx}^{(2)}$ or $\rho_{yy}^{(1)}/\rho_{yy}^{(2)}$.

For the case (i), since ρ_{xx} is larger than ρ_{yy} current wants to flow to y direction more than in the case of (ii). So the current which entered in plate goes longer path, and as a result effectively it feels more resistance. Therefore, considering such phenomena, R_1/R_2 is bigger than $\rho_{xx}^{(1)}/\rho_{xx}^{(2)}$ which we just consider x-direction.

(i)



(ii)



Problem 3.

$$\vec{j} = \sigma(\vec{E} + \vec{v}\times\vec{B}), \quad \vec{B} = (0, 0, B)$$

$$j_i = \sigma(E_i + \epsilon_{ijk} v_j B_k), \quad \vec{j} = ne\vec{v} \Rightarrow v_i = \frac{j_i}{ne}$$

$$\begin{cases} j_1 = \sigma E_1 + \sigma\epsilon_{123} v_2 B \\ j_2 = \sigma(E_2 + \epsilon_{213} v_1 B) \end{cases}$$

$$\downarrow$$

$$\begin{cases} j_1 = \sigma E_1 + \frac{\sigma B}{ne} j_2 \\ j_2 = \sigma E_2 - \frac{\sigma B}{ne} j_1 \end{cases}$$

$$\downarrow$$

$$j_1 = \sigma E_1 + \frac{\sigma B}{ne}\left(\sigma E_2 - \frac{\sigma B}{ne} j_1\right)$$

$$j_1\left(1 + \left(\frac{\sigma B}{ne}\right)^2\right) = \sigma\left(E_1 + \frac{\sigma B}{ne} E_2\right)$$

$$j_1 = \frac{1\cdot\sigma}{1 + \left(\frac{\sigma B}{ne}\right)^2}\left(E_1 + \frac{\sigma B}{ne} E_2\right)$$

$$j_2 = \sigma E_2 - \frac{\sigma B}{ne} j_1 = \sigma E_2 - \frac{\sigma}{1 + \left(\frac{\sigma B}{ne}\right)^2}\left(\frac{\sigma B}{ne} E_1 + \frac{\sigma B}{ne} E_2\right) =$$

$$= \frac{\sigma}{1 + \left(\frac{\sigma B}{ne}\right)^2}\left(E_2 - \frac{\sigma B}{ne} E_1\right)$$

$$\Rightarrow \begin{pmatrix} j_1 \\ j_2 \end{pmatrix} = \frac{\sigma}{1 + \left(\frac{\sigma B}{ne}\right)^2}\begin{pmatrix} 1 & \frac{\sigma B}{ne} \\ -\frac{\sigma B}{ne} & 1 \end{pmatrix}\begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} = \frac{\sigma}{1 + \left(\frac{\sigma B}{ne}\right)^2}\begin{pmatrix} 1 & \frac{\sigma B}{ne} \\ -\frac{\sigma B}{ne} & 1 \end{pmatrix}.$$

3. $$\sigma_{xx} = \frac{\sigma}{1+\left(\frac{\sigma B}{ne}\right)^2}$$

$$\begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{xy} \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix}^{-1} = \begin{pmatrix} 1/\sigma & -\frac{B}{ne} \\ \frac{B}{ne} & 1/\sigma \end{pmatrix}$$

$\rho_{xx} = 1/\sigma$, If we fix B (finite ~~∞~~), let $\sigma \to \infty$

ρ_{xx}, σ_{xx} both $\to$ ~~∞~~ 0

$\rho_{xx} \to 0$ just because resistivity is inverse of conductivity

$\sigma_{xx} \to 0$ because when $\sigma \to 0$, from $\vec{J} = \sigma(\vec{E} + \frac{\vec{J}}{ne} \times \vec{B})$

$\Rightarrow \vec{E} + \frac{\vec{J} \times \vec{B}}{ne} \to 0 \Rightarrow \vec{J} \perp \vec{E} \Rightarrow$ diagonal components of

$\vec{\vec{\sigma}}$ must vanish.

4. For arbitrary B, ρ is an antisymmetric matric (from Prob 3)
$\rho_{xy} = -\rho_{yx}$, $\rho_{xx} = \rho_{yy}$

$$\begin{cases} E_x = \rho_{xx} k_x + \rho_{xy} k_y \\ E_y = \rho_{yx} k_x + \rho_{yy} k_y \end{cases} \qquad \nabla \times \vec{E} = 0 \Rightarrow \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} = 0$$

$$\Rightarrow \rho_{xx}\frac{\partial k_x}{\partial y} + \rho_{xy}\frac{\partial k_y}{\partial y} - \left(\rho_{yx}\frac{\partial k_x}{\partial x} + \rho_{yy}\frac{\partial k_y}{\partial x}\right) = 0$$

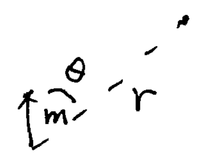
$$k_x = -\frac{\partial \phi}{\partial y}, \quad k_y = \frac{\partial \phi}{\partial x}$$

$$\Rightarrow \rho_{xx}\frac{\partial^2 \phi}{\partial y^2} + \rho_{yy}\frac{\partial^2 \phi}{\partial x^2} = 0 \Rightarrow \nabla^2 \phi = 0$$ (Equation doesn't depend on B)

$\Rightarrow$ Current flow pattern doesn't change with B.

5. In free space, we have

$$\vec{B}_0 = \frac{\mu_0 \vec{m}}{4\pi r^3} (2\cos\theta\, \hat{r} + \sin\theta\, \hat{\theta})$$



For linear magnetic materials, H doesn't change

$$\vec{B} = \frac{\vec{B}_0 \mu}{\mu_0} = \vec{B}_0 (1+\chi_m)$$

$$= \frac{\mu_0 (1+\chi_m)}{4\pi r^3} (2\cos\theta\, \hat{r} + \sin\theta\, \hat{\theta})$$