PROBLEM SET 3

1.

B=γt

L=30 cm

R=3 mm

γ=.001 T/s

ρ=1.6×10−6 ohm-meter

First, split the copper rod into rings, with width dr r and height dz. A given ring is located at a length z along the rod and at a radius r, with z and r varying from 0≤r≤R and 0≤z≤L. The flux through the ring is

Φ=Bπr^2

Due to the symmetry of the problem, the electrical field induced by the varying magnetic field should be constant around the ring. Thus, the line integral of the electrical field around the ring will just be the product of the field strength and ring circumference. Note for later that this line integral is literally the voltage around the ring. This gives

dΦ/dt=$\frac{dB}{dt}\pi$r^2=$\gamma\pi$r^2=E2πr=V

$\overrightarrow{E}$=(-rγ/2)$\hat{\theta}$

Using the previous week's derivation of the relation between σ, $\overrightarrow{E}$, and $\overrightarrow{B}$, and if we recognize that only the θ portion of the $\overrightarrow{J}$ will be relavent, we find that

J=$\sigma/(1+(\frac{tB}{ne})^2))(-\frac{\gamma}{2}\hat{\theta})$

$\sigma=1/\rho$

Using the relation that the power dissipated P is P=VI, and integrating θ from 0 to 2π, we find

P$_{loop}$=V$\int \overrightarrow{J}\cdot\hat{\theta}dA$=$\gamma^2\pi$r^{3}drdz/2$\rho(1+(\frac{tB}{ne})^2))$

The power dissipated in the entire rod is simply the integral of this quantity over all applicable z and r values. This gives

P=$\gamma^2\pi$Lr^4/8$\rho(1+(\frac{tB}{ne})^2))$
See the last page for the graph.
2.
a) $\nabla\times\overrightarrow{E}$=d$\overrightarrow{B}$/dt
Inside a perfect conductor, E is the 0 function on the microscopic level, so dB/dt is as well.

b) $\nabla\times\overrightarrow{E}$=d$\overrightarrow{B}$/dt
Consider a loop of perfectly conducting wire of radius a. The line intergral of the electrical field around the loop is equal to the time derivative of the magnetic flux in the loop. Since we have a perfect conductor, E = 0, the line intergral is 0, which requires the derivative of the flux to be 0, i.e. it is constant.

c)Since a superconductor is among other things a perfect conductor, the conductivity σ is infinite. A current of any magnitude can flow within the conductor with 0 $\overrightarrow{E}$ field, hence the electrical field must be 0.

d) As the sphere goes through it's superconducting transition, the superconductor must generate a current to cancel the B = Bo$\hat{z}$ field at each point inside the sphere, and that current must be confined to the surface of the sphere due to the arguments of the previous parts of the problem. Note from Ex 5.11 the magnetic field of a sphere with charge density σ, radius R, and rotating at an angular velocity ω generates a uniform magnetic field with the sphere of strength

$B=2\mu_0\sigma R\omega/3$

directed in the $+\hat{z}$ direction.

$\sigma=3B_o/(2\mu_0 R\omega)$

The current at an angle θ on the surface of the sphere is

$J=\sigma v=\sigma R\sin\theta\omega=3B_o\sin\theta/(2\mu_0)$

3.

a) $\overrightarrow{B}_{dipole}=\mu_0 m/4\pi r^3(2\cos\theta\hat{r}+\sin\theta\hat{\theta})$

The image dipole will point in the -z direction (if you are unsure, just remember the field perpendicular to the xy-plane at z=0 must be 0 at the point located in the center of the two dipoles).

b) The magnetic field the real dipole will experience is just that due to another dipole located at $2h$ below, but we will evaluate r explicitly later. In this case (θ=0)

$\overrightarrow{B}=2\mu_0 m/4\pi r^3(\hat{r})$

The force the real dipole experiences is $\nabla(\overrightarrow{B}\cdot\overrightarrow{m})$.

$\nabla(\overrightarrow{B}\cdot\overrightarrow{m})=6\mu_0 m^2/4\pi r^4$

Now explicitly evaluating r=2h gives us

$\overrightarrow{F}=Mg=3\mu_0 m^2/32\pi h^4$

$h=(3\mu_0 m^2/32Mg)^{1/4}$
c)

First, we will define the distance from the dipole as

$R^2=r^2+h^2$

To calculate the net magnetic field at the superconductor surface, we need to consider the field from both the image and the real dipoles, but we will focus on the real dipole first. We can then return to the general formula for the field from a magnetic dipole and substitute

$\hat{r}=\sin\theta\cos\psi\hat{x}+\sin\theta\sin\psi\hat{y}+\cos\theta\hat{z}$

and

$\hat{\theta}=\cos\theta\cos\psi\hat{x}+\cos\theta\sin\psi\hat{y}+\sin\theta\hat{z}$

Since the problem is rotationally symmetric, we can consider one particular ψ value (in this case ψ=0). This simplifies to

$\hat{r}$=sin$\theta\hat{x}$ + cos$\theta\hat{z}$

$\hat{\theta}$=cos$\theta\hat{x}$ + sin$\theta\hat{z}$

So far, we have

$\overrightarrow{B}_{real}$=$\mu_0$m/4$\pi$R^3((2cos$\theta$sin$\theta$+sin$\theta$cos$\theta$)$\hat{x}$ + (1+(cosθ)^2)$\hat{z}$)

To consider the magnetic fields around the xy-plane (were θ is close to $\pi/2$), we will use a change of variables.

cosθ=sinα

sinθ=cosα

α=θ+$\pi/2$

We can also note the geometry of the problem

sinα=h/R

cosα=r/R

This gives us a total result from the real dipole of

$\overrightarrow{B}_{real}$=3$\mu_0$m/4$\pi$R^3(hr/R^2)$\hat{x}$ + (1+(sinα)^2)μ_0m/4πR$^3\hat{z}$

$\overrightarrow{B}_{real}$=3$\mu_0$mhr/4$\pi$R$^5\hat{x}$ + (1+(sinα)^2)μ_0m/4πR$^3\hat{z}$

The image dipole will generate a similar magnetic field, but the angles it experiences are somewhat different, and the magnetic moment is opposite in sign with respect to the real dipole. This is manifest in the equations by h; the h for the image dipole is opposite in sign to the one for the real dipole, since the xy-plane is above the image but below the real dipole. The end result is we can calculate the magnetic field from the image dipole by letting h go to -h and m go to -m. Combining the results from the two fields, we see the magnetic field in the $\hat{x}$ direction additively combines, and the $\hat{z}$ components cancel.

$\overrightarrow{B}_{total}$=3$\mu_0$mhr/2$\pi$(r^2+h^2)^5/2$\hat{x}$

In this case, $\hat{y}$=$\hat{\psi}$, and we can see that for

$\overrightarrow{B}$=μ_0($\overrightarrow{K}$ × $\hat{z}$)

to be true,

$\overrightarrow{K}$=3$\mu_0$mhr/2π(r^2+h^2)^5/2$\hat{y}$=$\mu_0$mhr/2π(r^2+h^2)^5/2$\hat{\psi}$

4.

For a and a$\ll$z, the distance from one line segment to another is essentially constant at r^2=(b^2+z^2). Since r is indepedent of all variables, the line integrals are simply the circumference of each loop. This gives

M$_{21}$=$\mu_0\pi$ba/(b^2+z^2)

b)

Note that in this case
$\overrightarrow{dl_1}$ = ($acos\phi$dϕ,asinϕdϕ,0)

$\overrightarrow{dl_2} = (bcos\theta$d$\theta$,bsin$\theta$d$\theta$,0)

The generalized Neumann formula in this case is

$$M_{21}=(\mu_0\pi)\oint_0^{2\pi}\oint_0^{2\pi}\mathrm{ab}(\sin\theta\sin\phi+\cos\theta\cos\phi)\mathrm{d}\theta\mathrm{d}\phi/(\mathrm{z}^2+(\mathrm{b}\cos\theta+\mathrm{a}\cos\phi)^2+(\mathrm{b}\sin\theta+\mathrm{a}\sin\phi)^2)^{1/2}$$

The method of calculating by hand involves use of an elliptical integral expanded in a power series, but students are encouraged to computer based symbolic integration.

$$M=(\mu_0\pi\beta/2)(\mathrm{ab}\beta)^{1/2}(1+15/8\beta^2...)$$

$$\beta=(\mathrm{ab})/(\mathrm{z}^2+\mathrm{a}^2+\mathrm{b}^2)$$

Problem 5

a)

For the upper plate only, $\mathbf{B} = -(\mu_0/2)K\hat{x}$ above the plane, and $\mathbf{B} = +(\mu_0/2)K\hat{x}$ below the plane. For the lower plate only, $\mathbf{B} = +(\mu_0/2)K\hat{x}$ above the plane, and $\mathbf{B} = -(\mu_0/2)K\hat{x}$ below the plane. So together we have

$$\mathbf{B} = \begin{cases} \mu_0 K\hat{x} & \text{between the plates} \\ 0 & \text{everywhere else} \end{cases}$$

b)

First choose our coordinate system such that the upper plate is at $y = d/2$ and the lower plate is at $y = d/2$. The induced electric field satisfies:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu_0 \alpha \hat{x}$$

In analogy with Maxwell's eqn $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$, we can take $-\alpha\hat{x}$ to be the "body current" that generate $\mathbf{E}$. Furthermore, if we think of this "body current" as composed of stack of parallel infinite planes with surface current $\mathbf{K}'$, we can conclude that $\mathbf{E}$ has only $\hat{z}$ component , is antisymmetric for $z = 0$ plane, is in $-\hat{z}$ direction for $z > 0$ and in $\hat{z}$ direction for $z < 0$.

To calculate the electric field between the plates, draw a rectangle perpendicular to $\mathbf{B}$ and symmetrically cross $z = 0$ plane, we have:

$$\begin{aligned} & \oint \mathbf{E} \cdot d\mathbf{l} = \int\int -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a} \\ \Rightarrow \quad & 2Ez = \mu_o \alpha z 2y \\ \Rightarrow \quad & E = \mu_0 \alpha y \end{aligned} \tag{1}$$

To calculate the electric field outside the plates, let the rectangle cross the two plates, we then have:

$$\begin{aligned} & 2Ez = \mu_o \alpha z d \\ \Rightarrow \quad & E = \mu_0 \alpha d/2 \end{aligned} \tag{2}$$

c)

Outside the plates the Poynting vector is zero because there is no magnetic field. Between the plates:

$$\begin{aligned} \mathbf{S} &= \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \\ &= -\mu_0\alpha^2 yt\hat{y} \end{aligned} \tag{3}$$

The flux into the region between the plates is then

$$-S(z=d/2)+S(z=-d/2)=\mu_0\alpha^2 dt$$

The magnetic energy per unit area stored between the plates are:

$$\begin{aligned} E &= \frac{1}{2\mu_0}B^2 d \\ &= \mu_0\alpha^2 t^2 d/2 \end{aligned} \tag{4}$$

$\mathbf{E}$ is constant, so it won't contribute to the energe change rate. The energe change rate is then:

$$\frac{dE}{dt}=\mu_0\alpha^2 td \tag{5}$$

We see the flux matches the energy change rate.

d)

Here we can simply change $\alpha \rightarrow \partial K/\partial t$ and get:

$$\mathbf{B}=\begin{cases} \mu_0 K\hat{x} & \text{between the plates} \\ 0 & \text{everywhere else} \end{cases}$$

$$\mathbf{E}=-\mu_0\frac{\partial K}{\partial t}y\hat{z} \quad \text{between the plates}$$

$$\mathbf{E}=\pm\frac{\mu_0 d}{2}\frac{\partial K}{\partial t}\hat{z} \quad \text{outside the plates, - for } z>0$$

$$\mathbf{S}=\begin{cases} -\mu_0 K\frac{\partial K}{\partial t}y\hat{z} & \text{between the plates} \\ 0 & \text{everywhere else} \end{cases}$$

The flux into the region between the plates is then

$$-S(z=d/2)+S(z=-d/2)=\mu_0 K\frac{\partial K}{\partial t}d$$

The magnetic energy per unit area stored between the plates are:

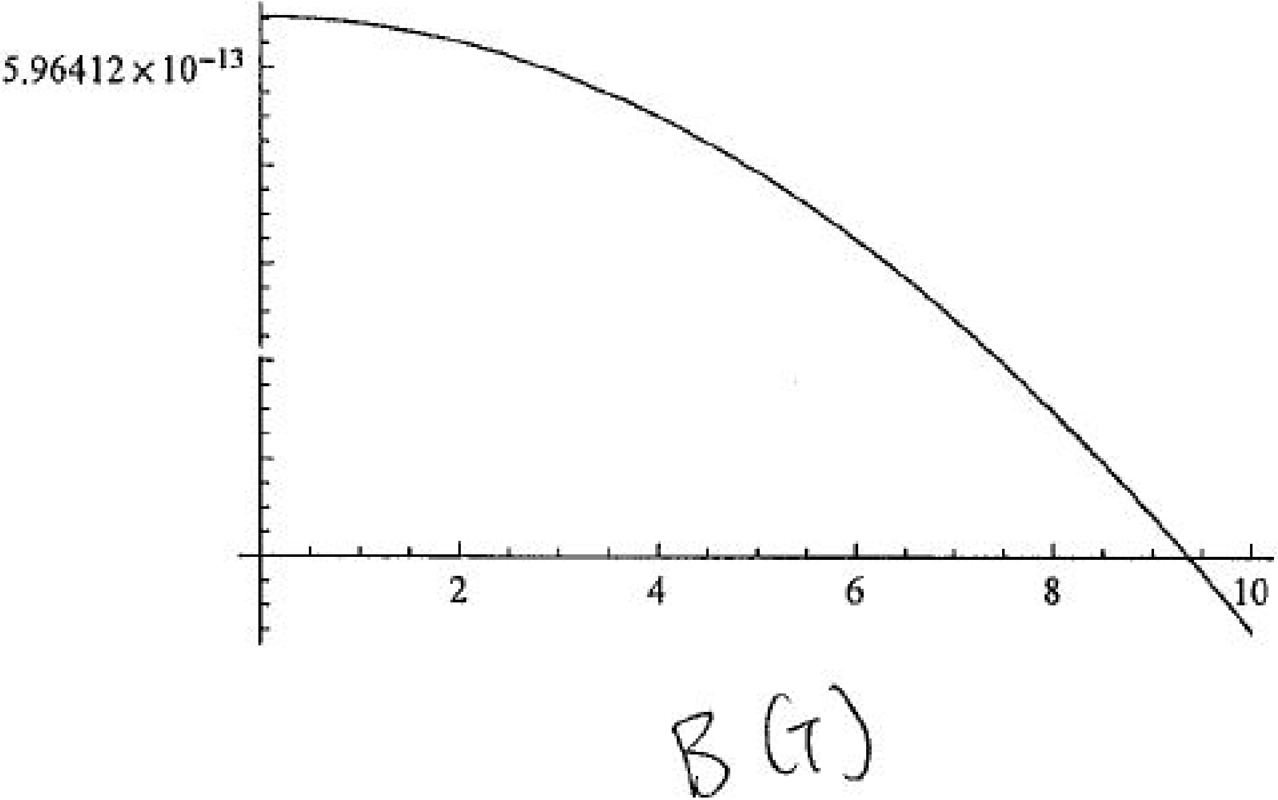
$$\begin{aligned} E &= \frac{1}{2\mu_0}B^2 d \\ &= \mu_0 K^2 d/2 \end{aligned} \tag{6}$$

The magnetic energe change rate is then:

$$\frac{dE}{dt} = \mu_0 K \frac{\partial K}{\partial t} d \tag{7}$$

We see the flux matches the magnetic energy change rate only.

This does not mean that total energy is not conserved, it's just the consequence of quasistaic approximation, where we can ignore the electric energy compared to the magnetic energy if the current variation frequency is very low. Total energy is always conserved. When the current variation is so fast that the electric energy is comparable to the magnetic energy, the quasistatic approximation breaks down and we need to take into account the displacement current, which infact reduces the original magnetic field.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