PROBLEM SET 5

1.

First, we need to consider the case where the two conductors are perfect conductors. We will model the parallel plate conductors as a series of small sections of length dl and consider a section with the ends located at l and l+dl. The capacitance and inductance between neighboring sections of the two plates are

dL=$\mu_0$ddl/W

dC=ϵ_0Wdl/d

The change in the voltage across a section is then given by

V(l+dl)=V(l)-iωIdL

V(l+dl)-V(l)$\simeq$dV=-i$\omega\mu_0$Iddl/W

dV/dl=-i$\omega\mu_0$dI/W

In a similar fashion, the change in the current along the length must take account of the capacitor.

dI=-iωdCV

dI/dl=-i$\omega\epsilon_0$W/d

Simply choose one equation, take the derivative with respect to dl.

d$^2 I/dl^2 + I\omega^2\epsilon_0\mu_0$=0

Now we consider the case where the conductivity of one conductor is dominated by the kinetic inductance of the electrons ($\omega\tau\gg1$). In this case

σ=(ne$^2\tau$/m)/(1-i$\omega\tau$)$\simeq$ine^2/mω

ρ=1/σ=-i(m/ne^2)ω

Now consider the conductivity of the imperfectly conducting plate. Taking account of the number of carriers, and the width and length of the strip, we find:

V=Iρdl/W=-iI(m/ne^2)ωdl/W

Note the similarity to the above voltage drop associated with the inductor, particularily the ω dependence. Thus, we can lump the two terms together in our derivation of the differential equation for I.

The ordinary magnetic field inductance will be negligible when

(m/ne^2)$\gg\mu_0$

The characteristic impedance of the line is

dV/dl=-iωLI

dI/dl=-iωCV

$V/I=L/C^*(dI/dl)/(dV/dl)$

However

$dI/dl=-i\omega(LC)^{1/2}I=dV/dl(C/L)^{1/2}$

$V/I=(L/C)^1/2=((m/ne^2W)/(\epsilon_0W/d))^{1/2}=(md/(ne^2\epsilon_0W^2))^{1/2}$

2.

See the final pages for a derivation of $\overrightarrow{B}$ and $\overrightarrow{E}$. Note that c in this problem has been relabeled d, and oriented along the z-axis. Then a=.5 cm, b = .5 cm, and d = 1.0 cm. For the lowest mode, n=1, and either l=1 and m=0, or l=0 and m=1 (we will choose l=1, m=0). We neglect l=0 and m=0 because it is a trivial solution.

$\omega=\frac{c\pi}{ad}(a^2+d^2)^{1/2}$

$E_x=0$

$E_y=(iB_0\omega a/\pi)\sin(\pi x/a)\sin(\pi z/d)e^{-it\omega}$

$E_z=0$

$B_x=(-B_0a/d)\sin(\pi x/a)\cos(\pi z/d)e^{-it\omega}$

$B_y=0$

$B_z=(B_0)\cos(\pi x/a)\sin(\pi z/d)e^{-it\omega}$

The energy stored in the mode is given by the integral of B^2 and E^2 over the entire resonator volume and average over time.

$U=1/2\int \epsilon_0E^*E + B^*B/\mu_0$

$U=(1/2){B_0}^2(\epsilon_0(\frac{a\omega}{\pi})^2+(a^2/d^2+1)/(\mu_0))^*(abd/4)$

$U=(1/2){B_0}^2(\epsilon_0c^2\frac{a^2+d^2}{d^2}+(a^2/d^2+1)/(\mu_0))^*(abd/4)$

$U=(1/2)abd{B_0}^2(\frac{a^2+d^2}{d^2})/(\mu_0))$

As per the class notes, the energy loss at a conducting surface is given by

$\overrightarrow{J}\cdot\overrightarrow{E}=K^2/(\delta^2\sigma)$

$\delta\simeq(2/\mu_0\omega\sigma)^{1/2}$

$\overrightarrow{K}\times\hat{n}=\overrightarrow{B}/\mu_0$

To consider the loss in the cavity, we must consider the surfaces with a paralell magnetic field; in this case, x=0, x=a, y=0, y=b, z=0, and z=d. We need to average the over the entire area of each side, up to the penetration depth, and over time as well.

I. x=0,a

$\int K^2=\int B_z^2/\mu_0=B_0^2bd\delta/4$

$\overrightarrow{J}\cdot\overrightarrow{E}=K^2/(\delta^2\sigma)=(B_0^2bd/4\mu\sigma\delta)$

For 2 sides, the power dissipated will be doubled.

$P_x=(B_0/(\mu))^2 bd/2\mu\sigma\delta)$

II. y=0,b

We need to perform similar calculation as above, except accounting for both B_z and B_x. The result is

$P_y=(B_0/(\mu))^2 ad/2\mu\sigma\delta)(1+(a^2)/(d^2))$

III. z=0,d

We solely need to account for B_x.

$P_z=((B_0/(\mu))^2 ab/2\mu\sigma\delta)(a^2)/(d^2))$

The total power dissipated is then

$P_t=(B_0{}^2/2\sigma\mu_0\delta))(bd + ad + a^3/d + a^3b/d^2)$

The quality factor is given by the total energy over the power dissipated over one period, or

$Q=\omega U/P_t=abd(1+d^2/a^2)/\delta(bd + ad + a^3/d + a^3b/d^2)$

$Q \sim 3*10^4$

Problem 5 G9.38

Putting boundaries in the z direction means that instead of propagating waves, there will be standing waves in that direction just like in the x and y directions. Thus the e^{ikz} in the expressions for the electric and magnetic fields in Eqn. 9.176 in Griffiths will have to be modified to $Ae^{ikz} + Be^{-ikz}$. Now, we know that the boundary conditions at the two ends in the z direction are

$$E_x = E_y = 0$$

and

$$B_z = 0$$

This immediately tells us that the z dependence of these quantities is of the form $\sin(kz)$ where

$$k = \frac{n\pi}{d}$$

for positive integers n. We can now write down the general expressions for the various components of the electric and magnetic fields inside the cavity. The electric field components are

$$\begin{aligned}
\tilde{E}_x(x,y,z,t) &= E_x(x,y)\sin(kz)e^{-i\omega t} \\
\tilde{E}_y(x,y,z,t) &= E_y(x,y)\sin(kz)e^{-i\omega t} \\
\tilde{E}_z(x,y,z,t) &= E_z(x,y)f_z(z)e^{-i\omega t}
\end{aligned}$$

and the magnetic field components are

$$\begin{aligned}
\tilde{B}_x(x,y,z,t) &= B_x(x,y)g_x(z)e^{-i\omega t} \\
\tilde{B}_y(x,y,z,t) &= B_y(x,y)g_y(z)e^{-i\omega t} \\
\tilde{B}_z(x,y,z,t) &= B_z(x,y)\sin(kz)e^{-i\omega t}
\end{aligned}$$

The dimensionless functions $f_z(z), g_x(z)$ and $g_y(z)$ are all of the form $Ae^{ikz} + Be^{-ikz}$ and have to be determined from Maxwell's equations. Putting these into Maxwell's equations (iii) and (iv) of 9.177 in Griffiths, we get

$$\frac{\partial E_y(x,y)}{\partial x} - \frac{\partial E_x(x,y)}{\partial y} = i\omega B_z(x,y)$$

$$\frac{\partial E_z(x,y)}{\partial y} - k\cos(kz)E_y(x,y) = i\omega B_x(x,y)g_x(z)$$

$$-\frac{\partial E_z(x,y)}{\partial x} + k\cos(kz)E_x(x,y) = i\omega B_y(x,y)g_y(z)$$

$$\frac{\partial B_y(x,y)}{\partial x}g_y(z) - \frac{\partial B_x(x,y)}{\partial y}g_x(z) = -i\frac{\omega}{c^2}E_z f_z(z)$$

$$\frac{\partial B_z(x,y)}{\partial y}\sin(kz) - \frac{dg_y(z)}{dz}B_y(x,y) = -i\frac{\omega}{c^2}E_x\sin(kz)$$

$$-\frac{\partial B_z(x,y)}{\partial x}\sin(kz) + \frac{dg_x(z)}{dx}B_x(x,y) = -i\frac{\omega}{c^2}E_y\sin(kz)$$

These equations are the resonant-cavity-equivalents of Eqn. 9.179 for wave guides in Griffiths and have to be hold for all x, y and z inside the cavity. This immediately tell us from the second, and

$$B_z = 0$$

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These equations are the resonant-cavity-equivalents of Eqn. 9.179 for wave guides in Griffiths and have to be hold for all x, y and z inside the cavity. This immediately tell us from the second, which along with the above equations gives us

$$\tilde{E}_x(x,y,z,t) = \frac{-iB_0\omega_{lmn}(m\pi/b)}{(l\pi/a)^2 + (m\pi/b)^2}\cos(l\pi x/a)\sin(m\pi y/b)\sin(n\pi z/d)e^{-i\omega t}$$

$$\tilde{E}_y(x,y,z,t) = \frac{iB_0\omega_{lmn}(l\pi/a)}{(l\pi/a)^2 + (m\pi/b)^2}\sin(l\pi x/a)\cos(m\pi y/b)\sin(n\pi z/d)e^{-i\omega t}$$

$$\tilde{B}_x(x,y,z,t) = \frac{-B_0(l\pi/a)(n\pi/d)}{(l\pi/a)^2 + (m\pi/b)^2}\sin(l\pi x/a)\cos(m\pi y/b)\cos(n\pi z/d)e^{-i\omega t}$$

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