Collective Effects
in
Equilibrium and Nonequilibrium Physics

Website: http://cnrs.bnu.edu.cn/mccross/Course/
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Today’s Lecture: Nonlinear Theory of Patterns near Onset

Outline

- Review: linear instability towards patterns
- Qualitative picture of nonlinear, spatially periodic patterns
- General Patterns Near Onset
  - One dimensional amplitude equation
  - Generalizations to two dimensions

Analogies to and differences from equilibrium phase transition to broken symmetry state
Review of Rayleigh-Bénard Instability
Linear Stability Analysis

- Driving strength: Rayleigh number $R \propto \Delta T$
- Look for linear mode $u, \theta \propto e^{\sigma(q)t} \cos(qx)$
- Calculate $\sigma(q)$ as a function of $R$
- $\sigma(q) > 0$ indicates exponential growth, i.e., instability towards a pattern with periodicity $2\pi/q$
Rayleigh’s Growth Rate (for $\mathcal{P} = 1$)

$R = 0.5 R_c$

$R_c = \frac{27\pi^4}{4}$, $q_c = \frac{\pi}{\sqrt{2}}$
Rayleigh’s Growth Rate (for $\mathcal{P} = 1$)

\[ R_c = \frac{27\pi^4}{4}, \quad q_c = \frac{\pi}{\sqrt{2}} \]
Rayleigh’s Growth Rate (for $P = 1$)

\[ \sigma = \frac{q}{\pi} \]

\[ R_c = \frac{27\pi^4}{4}, \quad q_c = \frac{\pi}{\sqrt{2}} \]
Rayleigh’s Growth Rate (for $\mathcal{P} = 1$)

\[ R_c = \frac{27\pi^4}{4}, \quad q_c = \frac{\pi}{\sqrt{2}} \]
Parabolic approximation near maximum

For $R$ near $R_c$ and $q$ near $q_c$

$$\text{Re } \sigma(q) = \tau_0^{-1} \left[ \varepsilon - \xi_0^2 (q - q_c)^2 \right] \quad \text{with} \quad \varepsilon = \frac{R - R_c}{R_c}$$
Neutral stability curve

\[ \text{Re } \sigma_q > 0 \]

\[ \text{Re } \sigma_q < 0 \]

\[ R_c \]

\[ q_c \]

Re \( \sigma (q) = 0 \) defines the neutral stability curve \( R = R_c (q) \) or \( q = q_N (R) \)

Rayleigh: \[ R_c (q) = \frac{(q^2 + \pi^2)^3}{q^2} \]
Linear Stability Analysis

Linear stability theory is often a useful first step in understanding pattern formation:

• Often is quite easy to do either analytically or numerically
• Displays the important physical processes
• Gives the length scale of the pattern formation $1/q_c$
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But:

• Leaves us with unphysical exponentially growing solutions
Linear Stability Analysis

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Nonlinear Theory

- Saturation of spatially periodic solution (bifurcation theory)
- General patterns (cf., broken symmetry at phase transitions)
Qualitative Picture of Nonlinear States

\[ \text{Re } \sigma_q > 0 \]

\[ \text{Re } \sigma_q < 0 \]
Qualitative Picture of Nonlinear States

\[ R_c(q) \] or \[ q_N(R) \]

\[ \text{Re } \sigma_q > 0 \]

\[ R_c \]

\[ q_c \]

\[ \text{Re } \sigma_q < 0 \]
Qualitative Picture of Nonlinear States

\[ R_c(q) \text{ or } q_N(R) \]

Re \[ \sigma_q > 0 \]

Re \[ \sigma_q < 0 \]

Band of growing solutions
Qualitative Picture of Nonlinear States: Periodic BC

\[ R = n \frac{2\pi}{l} \]

Forward Bifurcation
Qualitative Picture of Nonlinear States: Periodic BC
Qualitative Picture of Nonlinear States: Infinite System

\[ R \]

\[ R_c \]

\[ q_c \]

\[ q \]

nonlinear states
Qualitative Picture of Nonlinear States

Patterns exist. Are they stable?

No patterns
Qualitative Picture of Nonlinear States: Instability of Stripes

\[ E = \text{Eckhaus} \]
Qualitative Picture of Nonlinear States: Instability of Stripes

Z = ZigZag

unstable

stable

R

R_c

q_c

q
Qualitative Picture of Nonlinear States: Instability of Stripes
Qualitative Picture of Nonlinear States: Stability Balloon

E=Eckhaus
Z=ZigZag
SV=Skew Varicose
O=Oscillatory

stable band
Qualitative Picture of Nonlinear States: Stability Balloon

E= Eckhaus
Z= Zig Zag
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Tools for the Nonlinear Problem

• The instability to a pattern is another example of a broken symmetry transition, now in the context of nonequilibrium systems.

• The same basic ideas we discussed in the context of equilibrium phase transitions apply:
  ◦ near the transition \( R \simeq R_c, q \simeq q_c \) or slow modulations of a pattern at \( q_c \) describe the behavior using an order parameter
  ◦ away from the transition use a phase variable description to describe the behavior resulting from the broken symmetry

• There will be similar general behavior:
  ◦ new rigidity
  ◦ Goldstone modes
  ◦ importance of topological defects

• There will be important differences in formulation and behavior because we cannot start from a free energy, but must consider directly the dynamics
Amplitude Equations

Systematic approach for describing weakly nonlinear solutions near onset for solutions near a stripe state
**Amplitude Equations**

Linear onset solution for stripes

\[ \delta u_q(x_\perp, z, t) = \left[ a_0 e^{i(q-q_c) \cdot x_\perp} e^{Re \sigma q t} \right] \times \left[ u_q(z) e^{i q_c \cdot x_\perp} \right] + \text{c.c.} \]

Small terms near onset \hspace{2cm} \text{Onset solution}
Amplitude Equations

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Small terms near onset

Onset solution

Weakly nonlinear, slowly modulated, solution

\[ \delta u(x_\perp, z, t) \approx A(x_\perp, t) \times \left[ u_{q_c}(z) e^{i q_c \cdot x_\perp} \right] + \text{c.c.} \]

Complex amplitude

Onset solution
**Amplitude Equations**

Linear onset solution for stripes

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Small terms near onset  

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Complex amplitude  

Onset solution

$A(x_\perp, t)$ is the order parameter for the stripe state
**Amplitude Equations**

Linear onset solution for stripes

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Complex amplitude  
Onset solution

\(A(x_\perp, t)\) is the order parameter for the stripe state

\(A(x_\perp, t)\) satisfies the **amplitude equation**.
Amplitude Equations

Linear onset solution for stripes

\[
\delta u_q(x_\perp, z, t) = \left[ a_0 e^{i(q-q_c) \cdot x_\perp} e^{\text{Re} \sigma_q t} \right] \times \left[ u_q(z) e^{i q_c \cdot x_\perp} \right] + \text{c.c.}
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Weakly nonlinear, slowly modulated, solution

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\delta u(x_\perp, z, t) \approx A(x_\perp, t) \times \left[ u_q(z) e^{i q_c \cdot x_\perp} \right] + \text{c.c.}
\]

Complex amplitude

Onset solution

\(A(x_\perp, t)\) is the order parameter for the stripe state

\(A(x_\perp, t)\) satisfies the amplitude equation. In 1d [\(q_c = q_c \hat{x}, A = A(x, t)\)]:

\[
\tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \quad \varepsilon = (R - R_c)/R_c
\]
Complex Amplitude

Magnitude and phase of $A$ play very different roles

$$A(x, y, t) = a(x, y, t)e^{i\theta(x, y, t)}$$

$$\delta u(x_\perp, z, t) = ae^{i\theta} \times e^{iq_c x} u_{q_c}(z) + c.c.$$
Complex Amplitude

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- magnitude $a = |A|$ gives strength of disturbance
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- magnitude $a = |A|$ gives strength of disturbance
- phase change $\delta \theta$ gives shift of pattern (by $\delta x = \delta \theta/q_c$)
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- magnitude $a = |A|$ gives strength of disturbance

- phase change $\delta \theta$ gives shift of pattern (by $\delta x = \delta \theta / q_{c}$)— symmetry!
Complex Amplitude

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• magnitude $a = |A|$ gives strength of disturbance  
• phase change $\delta \theta$ gives shift of pattern (by $\delta x = \delta \theta / q_c$)—symmetry!  
• $x$-gradient $\partial_x \theta$ gives change of wave number $q = q_c + \partial_x \theta$  
  $$A = ae^{ikx}$$ corresponds to $q = q_c + k$
**Complex Amplitude**

Magnitude and phase of $A$ play very different roles

$$A(x, y, t) = a(x, y, t) e^{i\theta(x, y, t)}$$

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- phase change $\delta \theta$ gives shift of pattern (by $\delta x = \delta \theta / q_c$)— symmetry!
- $x$-gradient $\partial_x \theta$ gives change of wave number $q = q_c + \partial_x \theta$
  $$A = ae^{ikx} \text{ corresponds to } q = q_c + k$$
- $y$-gradient $\partial_y \theta$ gives rotation of wave vector through angle $\partial_y \theta / q_c$
  (plus $O[(\partial_y \theta)^2]$ change in wave number)
The amplitude equation describes

\[ \tau_0 \partial_t A = \varepsilon A - g_0 |A|^2 A + \xi_0^2 \partial_x^2 A \]

growth \hspace{1cm} saturation \hspace{1cm} dispersion/diffusion
Parameters

\[ \tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \]

• control parameter \( \varepsilon = (R - R_c)/R_c \)
Parameters

\[ \tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \]

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- system specific constants \( \tau_0, \xi_0, g_0 \)
Parameters

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\( \diamond \) \( \tau_0, \xi_0 \) fixed by matching to linear growth rate \( A = a e^{i k \cdot x_{\perp} e^{\sigma_q t}} \)
gives pattern at \( q = q_c \hat{x} + k \)

\[ \sigma_q = \tau_0^{-1} [\varepsilon - \xi_0^2 (q - q_c)^2] \]
Parameters

\[ \tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \]

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  - \( \tau_0, \xi_0 \) fixed by matching to linear growth rate \( A = a e^{i k \cdot x} e^{\sigma_q t} \)
    gives pattern at \( \mathbf{q} = \mathbf{q}_c \hat{x} + \mathbf{k} \)
    \[ \sigma_q = \tau_0^{-1} [\varepsilon - \xi_0^2 (q - q_c)^2] \]
  - \( g_0 \) by calculating nonlinear state at small \( \varepsilon \) and \( q = q_c \).
Scaling

\[ \tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \quad \varepsilon = \frac{R - R_c}{R_c} \]
Scaling

\[ \tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \quad \varepsilon = \frac{R - R_c}{R_c} \]

Introduce scaled variables

\[ x = \varepsilon^{-1/2} \xi_0 X \]

\[ t = \varepsilon^{-1} \tau_0 T \]

\[ A = (\varepsilon/g_0)^{1/2} \bar{A} \]
Scaling

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This reduces the amplitude equation to a *universal* form

\[ \partial_T \tilde{A} = \tilde{A} + \partial_x^2 \tilde{A} - |\tilde{A}|^2 \tilde{A} \]
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Since solutions to this equation will develop on scales \( X, Y, T, \bar{A} = O(1) \) this gives us scaling results for the physical length scales.
Derivation

\[ \tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \]

\[ \varepsilon = \frac{R - R_c}{R_c} \]
Derivation

\[ \tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \]
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• Expand dynamical equation in powers of \( A \) and use symmetry arguments (cf., equilibrium phase transitions where we expand free energy).
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- Expand dynamical equation in powers of \( A \) and use symmetry arguments (cf., equilibrium phase transitions where we expand free energy). Equation must be invariant under:
  - \( A(x) \to A(x)e^{i\Delta} \) with \( \Delta \) a constant, corresponding to a physical translation
  - \( A(x) \to A^*(-x) \), corresponding to inversion of the horizontal coordinates (parity symmetry)
Derivation

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  - \( A(x) \rightarrow A^*(-x) \), corresponding to inversion of the horizontal coordinates (parity symmetry)
- Multiple scales perturbation theory (Newell and Whitehead, Segel 1969)
- Mode projection (MCC 1980)
Amplitude Equation = Ginzburg Landau equation

\[ \tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \]

Familiar from other branches of physics:

- Good: take intuition from there
- Bad: no really new effects

E.g. equation is relaxational (potential, Lyapunov)

\[ \tau_0 \partial_t A = -\frac{\delta V}{\delta A^*}, \quad V = \int dx \left[ -\varepsilon |A|^2 + \frac{1}{2} g_0 |A|^4 + \xi_0^2 |\partial_x A|^2 \right] \]

This leads to

\[ \frac{dV}{dt} = -\tau_0^{-1} \int dx |\partial_t A|^2 \leq 0 \]

And dynamics runs “down hill” to a minimum of \( V \) — no chaos!
• We have arrived at the same Landau type formulation with an effective “potential” or “free energy” $V$!

• This is not fundamental, and is “luck” resulting from our expansion in $\varepsilon$ to lowest order
  
  ◦ no effective potential at higher order
  
  ◦ no effective potential for some side-wall boundary conditions
  
  ◦ no effective potential for rotating convection (and there is chaos at onset!)
Applications

What we can calculate:

• Effect of distant sidewalls
• Eckhaus instability
• Propagation of pattern into no pattern region (e.g., from localized initial condition)
• Evolution from random initial condition
• …
Example: Effect of Distant Sidewalls

One dimensional geometry with sidewalls that suppress the pattern
(e.g. rigid walls in a convection system)

\[ \partial_T \tilde{A} = \tilde{A} + \partial_X^2 \tilde{A} - |\tilde{A}|^2 \tilde{A} \quad \tilde{A}(0) = 0 \]

\[ \tilde{A} = e^{i\theta} \tanh(X/\sqrt{2}) \]
Example: Effect of Distant Sidewalls

One dimensional geometry with sidewalls that suppress the pattern (e.g. rigid walls in a convection system)

\[
\partial_T \tilde{A} = \tilde{A} + \partial_X^2 \tilde{A} - |\tilde{A}|^2 \tilde{A} \quad \tilde{A}(0) = 0
\]

Unscaled variables:

\[
\tilde{A} = e^{i\theta} \tanh(X/\sqrt{2})
\]

\[
A = e^{i\theta} \left(\epsilon/g_0\right)^{1/2} \tanh(x/\xi) \quad \text{with} \quad \xi = \sqrt{2\epsilon^{-1/2}} \xi_0
\]
Solution

\[ A = e^{i\theta} \left( \frac{\varepsilon}{g_0} \right)^{1/2} \tanh(\frac{x}{\xi}) \]

- suppression of pattern over length \( \varepsilon^{-1/2} \xi_0 \)
- arbitrary position of rolls
- asymptotic wave number is \( k = 0 \), giving \( q = q_c \): no band of existence
Solution

\[ A = e^{i\theta} (\varepsilon / g_0)^{1/2} \tanh(x / \xi) \]

- suppression of pattern over length \( \varepsilon^{-1/2} \xi_0 \)
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Extended amplitude equation to next order in \( \varepsilon \) (MCC, Daniels, Hohenberg, and Siggia 1980) shows

- discrete set of roll positions
- solutions restricted to a narrow \( O(\varepsilon^1) \) wave number band with wave number far from the wall

\[ \alpha_- \varepsilon < q - q_c < \alpha_+ \varepsilon \]
Electroconvection in a Smectic Film

Electroconvection in a Smectic Film

From Morris et al. (1991) and Mao et al. (1996)
Onset in Systems with Rotational Symmetry

- Two dimensional amplitude equation for stripes
- Amplitude equations for lattice states
- Rotationally invariant “model equation”
Stripe state
Rotational symmetry: amplitude equation for stripes

For a 2d, rotationally invariant system the gradient term is more complicated

$$\tau_0 \partial_t A = \varepsilon A + \xi_0^2 \left( \partial_x - \frac{i}{2q_c} \partial_y^2 \right)^2 A - g_0 |A|^2 A$$

$$q - q_c = \sqrt{(q_c + Q_x)^2 + Q_y^2} - q_c \approx Q_x + \frac{Q_y^2}{2q_c}$$
Rotational symmetry: amplitude equation for stripes

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Note: the complex amplitude can only describe small reorientations of the stripes.
Rotational symmetry: amplitude equation for stripes

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Note: the complex amplitude can only describe small reorientations of the stripes.

Isotropic system gives anisotropic scaling: \(x = \varepsilon^{-1/2} \xi_0 X; y = \varepsilon^{-1/4} (\xi_0/q_c)^{1/2} Y\)
Hexagonal state
**Amplitude theory of hexagons**

Amplitudes of rolls at 3 orientations $A_i(r, t)$, $i = 1 \ldots 3$

\[
\begin{align*}
\frac{dA_1}{dt} &= \varepsilon A_1 - A_1(A_1^2 + gA_2^2 + gA_3^2) + \gamma A_2 A_3 \\
\frac{dA_2}{dt} &= \varepsilon A_2 - A_2(A_2^2 + gA_3^2 + gA_1^2) + \gamma A_3 A_1 \\
\frac{dA_3}{dt} &= \varepsilon A_3 - A_3(A_3^2 + gA_1^2 + gA_2^2) + \gamma A_1 A_2
\end{align*}
\]

- $A_1 \neq 0, A_2 = A_3 = 0$ gives stripes
- $A_1 = A_2 = A_3 \neq 0$ gives hexagons
**Amplitude theory of hexagons**

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- $A_1 \neq 0, A_2 = A_3 = 0$ gives stripes
- $A_1 = A_2 = A_3 \neq 0$ gives hexagons

For $A_i \rightarrow -A_i$ symmetry, $\gamma = 0$ and stripes v. hexagons depends on $g$

For no $A_i \rightarrow -A_i$ symmetry, $\gamma \neq 0$ and always get hexagons at onset
Swift-Hohenberg Equation

Rotationally invariant formulation in terms of a scalar field $\psi(x, y, t)$ that captures the same physics as the amplitude equation

$$\partial_t \psi = \left[ \varepsilon - (\nabla_{\perp}^2 + 1)^2 \right] \psi - \psi^3$$

$[\nabla_{\perp} = (\partial/\partial x, \partial/\partial y)]$
Swift-Hohenberg Equation

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- originally introduced to investigate *universal* aspects of the transition to stripes
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- later used to study qualitative aspects of stripe pattern formation
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• later used to study qualitative aspects of stripe pattern formation
• no systematic derivation: model rather than controlled approximation
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• originally introduced to investigate *universal* aspects of the transition to stripes
• later used to study qualitative aspects of stripe pattern formation
• no systematic derivation: model rather than controlled approximation
• equation is again relaxational

$$\partial_t \psi = -\frac{\delta V}{\delta \psi}, \quad V = \iint dxdy \left\{ -\frac{1}{2} \varepsilon \psi^2 + \frac{1}{2} \left[ (\nabla_\perp^2 + 1) \psi \right]^2 + \frac{1}{4} \psi^4 \right\}$$
Motivation

• Mode amplitude $\psi_q(t)$ at wave vector $q$ satisfies linear equation (for $q \simeq q_c$)

$$\dot{\psi}_q = \tau_0^{-1} [\epsilon - \xi_0^2 (q - q_c)^2] \psi_q$$

• To be able to write this as a local equation for the Fourier transform $\psi(x, y, t)$ approximate this by

$$\dot{\psi}_q = \tau_0^{-1} [\epsilon - (\xi_0^2 / 4q_c^2)(q^2 - q_c^2)^2] \psi_q$$

• In real space this gives

$$\tau_0 \dot{\psi}(x, y, t) = \epsilon \psi - (\xi_0^2 / 4q_c^2)(\nabla_\perp^2 + q_c^2)^2 \psi$$

Simplest linear pde that gives the ring of unstable modes (for $\epsilon > 0$)

• Add simplest possible nonlinear saturating term

$$\tau_0 \dot{\psi}(x, y, t) = \epsilon \psi - (\xi_0^2 / 4q_c^2)(\nabla_\perp^2 + q_c^2)^2 \psi - g_0 \psi^3$$

• Alternative motivation:

$$A(x, y)e^{iq_cx} \Rightarrow \psi(x, y)$$
Relaxation to steady state

(from Greenside and Coughran, 1984)
Coarsening in a periodic geometry

(From Elder, Vinals, and Grant 1992)
Generalized Swift-Hohenberg models

Qualitatively include other physics:
Generalized Swift-Hohenberg models

Qualitatively include other physics:

- break $\psi \rightarrow -\psi$ symmetry

$$\partial_t \psi = \left[ r - (\nabla_{\perp}^2 + 1)^2 \right] \psi + \gamma \psi^2 - \psi^3$$
Generalized Swift-Hohenberg models

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$$\partial_t \psi = \left[ r - (\nabla_\perp^2 + 1)^2 \right] \psi + \gamma \psi^2 - \psi^3$$

• change nonlinearity to make equation non-potential, e.g.

$$\partial_t \psi = \left[ r - (\nabla_\perp^2 + 1)^2 \right] \psi + (\nabla_\perp \psi)^2 \nabla_\perp^2 \psi$$
Generalized Swift-Hohenberg models

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$$\partial_t \psi = \left[ r - (\nabla_{\perp}^2 + 1)^2 \right] \psi + \gamma \psi^2 - \psi^3$$

- change nonlinearity to make equation non-potential, e.g.

$$\partial_t \psi = \left[ r - (\nabla_{\perp}^2 + 1)^2 \right] \psi + (\nabla_{\perp} \psi)^2 \nabla_{\perp}^2 \psi$$

- model effects of rotation

$$\partial_t \psi = \left[ r - (\nabla_{\perp}^2 + 1)^2 \right] \psi - \psi^3 +$$

$$g_2 \mathbf{\hat{z}} \cdot \nabla_{\perp} \times [ (\nabla_{\perp} \psi)^2 \nabla_{\perp} \psi ] + g_3 \nabla_{\perp} \cdot [ (\nabla_{\perp} \psi)^2 \nabla_{\perp} \psi ]$$
Conclusions

I have introduced the ideas and methods used to understand nonlinear patterns, focusing on the regime near threshold.

Next Lecture: Symmetry Aspects of Nonlinear Patterns

• Analogies with and differences from equilibrium phase transitions
• Phase variable description
• Topological defects