Pattern Formation in Spatially Extended Systems
Lecture 1

• some pictures
• linear instability
• nonlinear saturation
• stability balloon
• amplitude equation
Pattern formation

The formation of spatial structure in systems that are:
Pattern formation

The formation of spatial structure in systems that are:

• driven…
Pattern formation

The formation of spatial structure in systems that are:

• driven…

• dissipative…
Pattern formation

The formation of spatial structure in systems that are:

• driven…
• dissipative…
• therefore, nonequilibrium…
Pattern formation

The formation of spatial structure in systems that are:

- driven…
- dissipative…
- therefore, nonequilibrium…
- characterized by energy injection, transport, and dissipation…
Pattern formation

The formation of spatial structure in systems that are:

- driven…
- dissipative…
- therefore, nonequilibrium…
- characterized by energy injection, transport, and dissipation…
- and so cannot be described in terms of the minimization of a (free) energy
Microwave background from a 30° by 100° portion of the sky showing fluctuations of about $10^{-4} K$. (Boomerang project).
Angular power spectrum of Boomerang data.
Wind-swept sand at the Kelso dune field of the Mojave desert in California. The ripple spacing is about 10 cm.
Photomicrographs of snowflakes by Wilson Bentley.
Starving slime mold colony in the early stages of aggregation. The light regions correspond to cells that are moving with a speed of about 400 $\mu \text{min}^{-1}$ towards higher secretant concentrations (chemotaxis). [Figure from Florian Siegert].
From the website of G. Ahlers
Vertically shaken layer of fluid [From Kudrolli, Pier, and Gollub]
Vertically shaken layer of tiny balls (layer depth 1.2mm, ball diameter 0.15-0.18mm) [From Melo, Umbanhowar, and Swinney]
Equilibrium

Driving

R

Perturbations grow

R_c

Perturbations decay

Equilibrium
Equilibrium

Driving

\( R \)

\( R_c \)

Perturbations grow

Perturbations decay

Equilibrium
Pattern formation occurs in a spatially extended system when the growing perturbation about the spatially uniform state has spatial structure (a mode with nonzero wave vector).
Dynamical Equations

I shall confine my discussion to systems far from equilibrium that are macroscopic and continuous.

These are defined by dynamical equations that

- Reflect the laws of thermodynamics and the return to (local) equilibrium
- Are the familiar phenomenological equations

Leads us to the study of nonlinear, deterministic, PDEs
Equations for Convection (Boussinesq)

\[ \sigma^{-1} (\partial_t + \mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p + RT\hat{z} + \nabla^2 \mathbf{v} \]

\[ (\partial_t + \mathbf{v} \cdot \nabla) T = \nabla^2 T \]

\[ \nabla \cdot \mathbf{v} = 0 \]

Boundary conditions

\[ \mathbf{v} = 0 \quad \text{at} \quad z = 0, 1 \]

\[ T = \begin{cases} 
1 & \text{at} \quad z = 0 \\
0 & \text{at} \quad z = 1 
\end{cases} \]
Equations for Convection (Boussinesq)

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\]

\[
T = \begin{cases} 
1 & \text{at } z = 0 \\
0 & \text{at } z = 1 
\end{cases}
\]

Conducting (no pattern) solution: \( \mathbf{v} = 0, T = 1 - z \)
A first approach to patterns: linear stability analysis

1. Find equations of motion of the physical variables $u(x, y, z, t)$
2. Find the uniform base solution $u_b(z)$ independent of $x, y, t$
3. Focus on deviation from $u_b$

$$u(x, t) = u_b(z) + \delta u(x, t)$$

4. Linearize equations about $u_b$, i.e. substitute into equations of part (1) and keep all terms with just one power of $\delta u$. This will give an equation of the form

$$\partial_t \delta u = \hat{L} \delta u$$

where $\hat{L}$ may involve $u_b$ and include spatial derivatives acting on $\delta u$

5. Since $\hat{L}$ is independent of $x, y, t$ we can find solutions

$$\delta u_q(x_\perp, z, t) = u_q(z) e^{i q \cdot x_\perp} e^{\sigma_q t}$$
Exponential growth: $\exp[\sigma_q t]$

$$\delta u_q(x_\perp, z, t) = u_q(z) e^{iq \cdot x_\perp} e^{\sigma_q t}$$
Exponential growth: \( \exp[\sigma_q t] \)

\( \delta \mathbf{u}_q(x_\perp, z, t) = \mathbf{u}_q(z) e^{iq \cdot x_\perp} e^{\sigma_q t} \)

Re \( \sigma_q \) gives exponential growth or decay
Exponential growth: \( \exp[\sigma_q t] \)

\( \delta u_q(x'_\perp, z, t) = u_q(z) e^{i q \cdot x'_\perp} e^{\sigma_q t} \)

Re \( \sigma_q \) gives exponential growth or decay

Im \( \sigma_q = -\omega_q \) gives oscillations, waves \( e^{i(q \cdot x'_\perp - \omega_q t)} \)

\( \text{Im } \sigma_q = 0 \implies \text{Stationary instability} \)

\( \text{Im } \sigma_q \neq 0 \implies \text{Oscillatory instability} \)
Exponential growth: \( \exp[\sigma_q t] \)

\[
\delta u_q(x_{\perp}, z, t) = u_q(z) e^{iq \cdot x_{\perp}} e^{\sigma_q t}
\]

Re \( \sigma_q \) gives exponential growth or decay

Im \( \sigma_q = -\omega_q \) gives oscillations, waves \( e^{i(q \cdot x_{\perp} - \omega_q t)} \)

\[
\begin{align*}
\text{Im } \sigma_q &= 0 & \Rightarrow & \text{ Stationary instability} \\
\text{Im } \sigma_q \neq 0 & \Rightarrow & \text{ Oscillatory instability}
\end{align*}
\]

For this lecture I will look at the case of stationary instability
Rayleigh’s calculation

\[
\delta T_q(x, z) = (q^2 + \pi^2)^2 \cos(\pi z) \cos(qx), \\
\delta w_q(x, z) = q^2 \cos(\pi z) \cos(qx), \\
\delta u_q(x, z) = -i \pi q \sin(\pi z) \sin(qx).
\]
Rayleigh’s calculation

\[ (\sigma^{-1}\sigma_q + \pi^2 + q^2)(\sigma_q + \pi^2 + q^2) - Rq^2/(\pi^2 + q^2) = 0 \]
Parabolic approximation near maximum

For $R$ near $R_c$ and $q$ near $q_c$

$$\text{Re } \sigma_q = \tau_0^{-1} \left[ \varepsilon - \xi_0^2 (q - q_c)^2 \right]$$

with

$$\varepsilon = \frac{R - R_c}{R_c}$$
For $R$ near $R_c$ and $q$ near $q_c$

\[
\text{Re } \sigma_q = \tau_0^{-1} [\varepsilon - \xi_0^2 (q - q_c)^2] \quad \text{with} \quad \varepsilon = \frac{R - R_c}{R_c}
\]
Neutral stability curve

Rayleigh: \[ R_c(q) = \frac{(q^2 + \pi^2)^3}{q^2} \Rightarrow R_c = \frac{27\pi^4}{4}, \quad q_c = \frac{\pi}{\sqrt{2}} \]
Linear stability theory is often a useful first step in understanding pattern formation:

• Often is quite easy to do either analytically or numerically
• Displays the important physical processes
• Gives the length scale of the pattern formation $1/q_c$
**Linear stability theory** is often a useful first step in understanding pattern formation:

- Often is quite easy to do either analytically or numerically
- Displays the important physical processes
- Gives the length scale of the pattern formation $1/q_c$

But:

- Leaves us with unphysical exponentially growing solutions
Nonlinearity
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Tel Aviv, December, 2005:

\[
\text{Re } \sigma_q > 0 \\
\text{Re } \sigma_q < 0
\]
\[ \text{Re } \sigma_q > 0 \]

\[ \text{Re } \sigma_q < 0 \]

\[ R_c(q) \text{ or } q_N(R) \]
Tel Aviv, December, 2005: Pattern Formation in Spatially Extended Systems - Lecture 1

\[
R_c(q) \text{ or } q_{N}(R)
\]

\[\text{Re } \sigma_q > 0\]

\[\text{Re } \sigma_q < 0\]

\[q_{N-} \quad q_c \quad q_{N+}\]

band of growing solutions
Tel Aviv, December, 2005: *Pattern Formation in Spatially Extended Systems - Lecture 1*

Pattern formation in spatially extended systems involves understanding the dynamics of systems where spatial interactions lead to emergent patterns. One key aspect of these systems is the study of bifurcations, which are critical points where small changes in parameters can lead to significant qualitative changes in the system's behavior.

Mathematically, a bifurcation can be described by equations and diagrams that illustrate the transition from one state to another. The figure above shows a typical scenario involving a parameter $R$ and the bifurcation point $R_c$. The diagram illustrates a forward bifurcation, where the system transitions from one state to another as $R$ passes through $R_c$. The inset diagram specifically highlights the forward bifurcation, showing how a small change in $\delta u$ can lead to a significant change in the system's behavior.

The condition $q = n \frac{2\pi}{l}$ indicates a specific relationship between the wave number $q$ and the system's length scale $l$, which is crucial for understanding the patterns that emerge. This relationship is often seen in systems where periodic boundary conditions are applied.
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\[ R = \frac{n^2 \pi}{l} \]

Backward Bifurcation

\[ \delta u \]

\[ R \]

\[ q = n \frac{2\pi}{l} \]

\[ q_c \]

\[ R_c \]
nonlinear states
Patterns exist.  Are they stable?  

No patterns
Z=ZigZag

unstable

stable

q_c

R

R_c

q
E=Eckhaus
Z=ZigZag
SV=Skew Varicose
O=Oscillatory

stable band
R

Rc

qN-

qS-

qc

qS+

qN+

E=Eckhaus
Z=ZigZag
SV=Skew Varicose
O=Oscillatory

stable band
Tools for the Nonlinear Problem
Amplitude Equations
Systematic approach for describing weakly nonlinear solutions near onset
Linear onset solution

\[ \delta u_q(x, z, t) = \left[ a_0 e^{i(q-q_c) \cdot x} e^{\Re \sigma_q t} \right] \times \left[ u_q(z) e^{i q_c \cdot x} \right] + c.c. \]

Small terms near onset

Onset solution
Linear onset solution

$$\delta u_q(x_{\perp}, z, t) = \left[ a_0 e^{i(q - q_c) \cdot x_{\perp}} e^{\text{Re} \sigma_q t} \right] \times \left[ u_q(z) e^{i q_c \cdot x_{\perp}} \right] + c.c.$$  
Small terms near onset Onset solution

Weakly nonlinear, slowly modulated, solution

$$\delta u(x_{\perp}, z, t) \approx A(x_{\perp}, t) \times \left[ u_{q_c}(z) e^{i q_c \cdot x_{\perp}} \right] + c.c.$$  
Complex amplitude Onset solution
Linear onset solution

\[ \delta u_q(x_\perp, z, t) = \left[ a_0 e^{i(q - q_c) \cdot x_\perp} e^{\text{Re} \sigma_q t} \right] \times \left[ u_q(z) e^{i q_c \cdot x_\perp} \right] + c.c. \]

Small terms near onset \hspace{1cm} \text{Onset solution}

Weakly nonlinear, slowly modulated, solution

\[ \delta u(x_\perp, z, t) \approx A(x_\perp, t) \times \left[ u_{q_c}(z) e^{i q_c \cdot x_\perp} \right] + c.c. \]

Complex amplitude \hspace{1cm} \text{Onset solution}

Substituting into the dynamical equations gives the amplitude equation, which in 1d \([q_c = q_c \hat{x}, A = A(x, t)]\) is

\[ \tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \hspace{1cm} \varepsilon = \frac{R - R_c}{R_c} \]
Pictorially

A convection pattern that varies **gradually** in space

\[ u \propto \text{Re}[A(x)e^{iq_c x}] \]

\[ q_c = 3.117; \quad A(x) = 1 + 0.1 \cos(0.2x) \]
Complex Amplitude

Magnitude and phase of $A$ play very different roles

$$A(x, y, t) = a(x, y, t)e^{i\phi(x, y, t)}$$
Complex Amplitude

Magnitude and phase of $A$ play very different roles

$$A(x, y, t) = a(x, y, t)e^{i\theta(x, y, t)}$$

$$\delta u(x_\perp, z, t) = ae^{i\theta} \times e^{iq_c x} u_{q_c}(z) + \text{c.c.}$$
Complex Amplitude

Magnitude and phase of $A$ play very different roles

$$A(x, y, t) = a(x, y, t)e^{i\theta(x, y, t)}$$

$$\delta u(x_\perp, z, t) = a e^{i\theta} \times e^{iq_c x} u_{q_c}(z) + c.c.$$ 

- magnitude $a = |A|$ gives strength of disturbance
Complex Amplitude

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$$\delta u(x_\perp, z, t) = ae^{i\theta} \times e^{i q_c x} u_{q_c} (z) + c.c.$$ 

- magnitude $a = |A|$ gives strength of disturbance
- phase $\delta \theta$ gives shift of pattern (by $\delta x = \delta \theta / q_c$)
Complex Amplitude

Magnitude and phase of $A$ play very different roles

$$A(x, y, t) = a(x, y, t)e^{i\theta(x, y, t)}$$

$$\delta u(x_\perp, z, t) = ae^{i\theta} \times e^{iqc x} u_{qc}(z) + c.c.$$

- magnitude $a = |A|$ gives strength of disturbance
- phase $\delta \theta$ gives shift of pattern (by $\delta x = \delta \theta / q_c$) — symmetry!
Complex Amplitude

Magnitude and phase of $A$ play very different roles

\[ A(x, y, t) = a(x, y, t)e^{i\theta(x, y, t)} \]

\[ \delta u(x_\perp, z, t) = ae^{i\theta} \times e^{iq_c x} u_{q_c}(z) + c.c. \]

- magnitude $a = |A|$ gives strength of disturbance
- phase $\delta \theta$ gives shift of pattern (by $\delta x = \delta \theta / q_c$)—symmetry!
- x-gradient $\partial_x \theta$ gives change of wave number $q = q_c + \partial_x \theta$
  
  $A = ae^{ikx}$ corresponds to $q = q_c + k$
Complex Amplitude

Magnitude and phase of $A$ play very different roles

$$A(x, y, t) = a(x, y, t)e^{i\theta(x, y, t)}$$

$$\delta u(x_\perp, z, t) = ae^{i\theta} \times e^{iq_c x} u_{q_c}(z) + c.c.$$ 

- magnitude $a = |A|$ gives strength of disturbance
- phase $\delta \theta$ gives shift of pattern (by $\delta x = \delta \theta / q_c$)—symmetry!
- $x$-gradient $\partial_x \theta$ gives change of wave number $q = q_c + \partial_x \theta$
  $A = ae^{ikx}$ corresponds to $q = q_c + k$
- $y$-gradient $\partial_y \theta$ gives rotation of wave vector through angle $\partial_y \theta / q_c$
  (plus $O[(\partial_y \theta)^2]$ change in wave number)
The amplitude equation describes

$$\tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A$$

growth  dispersion/diffusion  saturation
Parameters

\[ \tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \]

• control parameter \( \varepsilon = (R - R_c)/R_c \)
Parameters

\[ \tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \]

- control parameter \( \varepsilon = (R - R_c)/R_c \)
- system specific constants \( \tau_0, \xi_0, g_0 \)
Parameters

\[ \tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \]

- control parameter \( \varepsilon = (R - R_c)/R_c \)
- system specific constants \( \tau_0, \xi_0, g_0 \)
  - \( \tau_0, \xi_0 \) fixed by matching to linear growth rate
  
  \[ A = a e^{i(k \cdot x - \sigma q t)} \] gives pattern at \( q = q_c \hat{x} + k \)

\[ \sigma_q = \tau_0^{-1} [\varepsilon - \xi_0^2 (q - q_c)^2] \]
Parameters

\[ \tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \]

- control parameter \( \varepsilon = (R - R_c)/R_c \)
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  - \( \tau_0, \xi_0 \) fixed by matching to linear growth rate
  \[ A = a e^{i k \cdot x} e^{\sigma_q t} \] gives pattern at \( q = q_c \hat{x} + k \)
  \[ \sigma_q = \tau_0^{-1} [\varepsilon - \xi_0^2 (q - q_c)^2] \]
  - \( g_0 \) by calculating nonlinear state at small \( \varepsilon \) and \( q = q_c \).
Scaling

\[ \tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \quad \varepsilon = \frac{R - R_c}{R_c} \]
Scaling

\[ \tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \quad \varepsilon = \frac{R - R_c}{R_c} \]

Introduce scaled variables

\[ x = \varepsilon^{-1/2} \xi_0 X \]
\[ t = \varepsilon^{-1} \tau_0 T \]
\[ A = (\varepsilon/g_0)^{1/2} \bar{A} \]
Scaling

\[ \tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \quad \varepsilon = \frac{R - R_c}{R_c} \]

Introduce scaled variables

\[ x = \varepsilon^{-1/2} \xi_0 X \]
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\[ A = (\varepsilon/g_0)^{1/2} \tilde{A} \]

This reduces the amplitude equation to a universal form

\[ \partial_T \tilde{A} = \tilde{A} + \partial_X^2 \tilde{A} - |\tilde{A}|^2 \tilde{A} \]
Scaling

\[ \tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \quad \varepsilon = \frac{R - R_c}{R_c} \]

Introduce scaled variables

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\[ A = (\varepsilon/g_0)^{1/2} \bar{A} \]

This reduces the amplitude equation to a universal form

\[ \partial_T \bar{A} = \bar{A} + \partial_x^2 \bar{A} - |\bar{A}|^2 \bar{A} \]

Since solutions to this equation will develop on scales \( X, Y, T, \bar{A} = O(1) \) this gives us scaling results for the physical length scales.
Derivation

\[ \tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \]

\[ \varepsilon = \frac{R - R_c}{R_c} \]
Derivation

\[ \tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \quad \varepsilon = \frac{R - R_c}{R_c} \]

- Symmetry arguments: equation invariant under:
  - \( A(x) \rightarrow A(x) e^{i \Delta} \) with \( \Delta \) a constant, corresponding to a physical translation;
  - \( A(x) \rightarrow A^*(-x) \), corresponding to inversion of the horizontal coordinates (parity symmetry);
Derivation

\[ \tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \quad \varepsilon = \frac{R - R_c}{R_c} \]

- Symmetry arguments: equation invariant under:
  - A. \( x \rightarrow x/\lambda \) with \( \lambda \) a constant, corresponding to a physical translation;
  - A. \( x \rightarrow -x \), corresponding to inversion of the horizontal coordinates (parity symmetry);
- Multiple scales perturbation theory (Newell and Whitehead, Segel 1969)
Derivation

\[ \tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \quad \varepsilon = \frac{R - R_c}{R_c} \]

- Symmetry arguments: equation invariant under:
  - \( A(\mathbf{x}_\perp) \rightarrow A(\mathbf{x}_\perp)e^{i\Delta} \) with \( \Delta \) a constant, corresponding to a physical translation;
  - \( A(\mathbf{x}_\perp) \rightarrow A^*(-\mathbf{x}_\perp) \), corresponding to inversion of the horizontal coordinates (parity symmetry);

- Multiple scales perturbation theory (Newell and Whitehead, Segel 1969)
- Mode projection (MCC 1980)
Amplitude Equation = Ginzburg Landau equation

\[ \tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \]

Familiar from other branches of physics:

- Good: take intuition from there
- Bad: no really new effects

e.g. equation is relaxational (potential, Lyapunov)

\[ \tau_0 \partial_t A = -\frac{\delta V}{\delta A^*}, \quad V = \int dx \left[ -\varepsilon |A|^2 + \frac{1}{2} g_0 |A|^4 + \xi_0^2 |\partial_x A|^2 \right] \]

This leads to

\[ \frac{dV}{dt} = -\tau_0^{-1} \int dx |\partial_t A|^2 \leq 0 \]

and dynamics runs “down hill” to a minimum of \( V \) — no chaos!
**Example:** one dimensional geometry with boundaries that suppress the pattern (e.g. rigid walls in a convection system)

First consider a single wall

\[ \partial_T \tilde{A} = \tilde{A} + \partial_X^2 \tilde{A} - |\tilde{A}|^2 \tilde{A} \quad \tilde{A}(0) = 0 \]

\[ \tilde{A} = e^{i\theta} \tanh(X/\sqrt{2}) \]

\[ A = e^{i\theta} (\varepsilon/g_0)^{1/2} \tanh(x/\xi) \quad \text{with} \quad \xi = \sqrt{2}\varepsilon^{-1/2}\xi_0 \]
\[ A = e^{i\theta} (\varepsilon / g_0)^{1/2} \tanh(x / \xi) \]

- arbitrary position of rolls
- asymptotic wave number is \( k = 0 \), giving \( q = q_c \): no band of existence
\[ A = e^{i\theta} (\varepsilon / g_0)^{1/2} \tanh(x / \xi) \]

- arbitrary position of rolls
- asymptotic wave number is \( k = 0 \), giving \( q = q_c \): no band of existence

Extended amplitude equation to next order in \( \varepsilon \) (MCC, Daniels, Hohenberg, and Siggia 1980) shows

- discrete set of roll positions
- solutions restricted to a narrow \( O(\varepsilon^1) \) wave number band with wave number far from the wall

\[ \alpha_- \varepsilon < q - q_c < \alpha_+ \varepsilon \]
Existence band

\[ \varepsilon \]

\[ E \]

\[ q_N \]

\[ q - q_c \]
From Morris et al. (1991) and Mao et al. (1996)
Mao et al. (1996)
Two sidewalls

\[ \xi \sim \varepsilon^{-1/2} \]
Conclusions

In today’s lectures I introduced some of the basic ideas of pattern formation:

- linear instability at nonzero wave number;
- nonlinear saturation;
- stability balloons.

I then introduced the amplitude equation which is the simplest theoretical approach that captures the key effects in pattern formation (growth, saturation, and dispersion).

I focussed on the equation in one dimension, and on a phenomenological derivation. You can find more technical aspects in the supplementary notes.

Next lecture: the role of continuous symmetries — rotation and translation