Pattern Formation in Spatially Extended Systems

Lecture 2: Symmetry

• Symmetry and stripes
  ◦ Rotational invariance near threshold
    ★ Amplitude equation
    ★ Swift-Hohenberg equation
  ◦ Translational invariance: the phase equation
    ★ Near threshold
    ★ Far from threshold
  ◦ Defects

• Lattice states
Rotational symmetry: linear instability
Rotational symmetry: amplitude equation for stripes

For a 2d, rotationally invariant system the gradient term is more complicated

$$\tau_0 \partial_t A = \varepsilon A + \xi_0^2 \left( \partial_x - \frac{i}{2q_c} \partial_y^2 \right)^2 A - g_0 |A|^2 A$$

$$q - q_c = \sqrt{(q_c + Q_x)^2 + Q_y^2} - q_c \approx Q_x + \frac{Q_y^2}{2q_c}$$
Rotational symmetry: amplitude equation for stripes

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Note: the complex amplitude can only describe small reorientations of the stripes.
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Isotropic system gives anisotropic scaling: $x = \varepsilon^{-1/2} \xi_0 X$; $y = \varepsilon^{-1/4} (\xi_0/q_c)^{1/2} Y$. 
Swift-Hohenberg equation

Simple equation for an order parameter $\psi(x, y, t)$ that is rotationally invariant in the plane and captures the same physics as the amplitude equation

$$\partial_t \psi = [r - (\nabla_\perp^2 + 1)^2] \psi - \psi^3$$
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- later used to study qualitative aspects of stripe pattern formation
- no systematic derivation: model rather than controlled approximation
- equation is relaxational

$$\partial_t \psi = -\frac{\delta V}{\delta \psi}, \quad V = \iint dxdy \left\{-\frac{1}{2} r \psi^2 + \frac{1}{2} \left[(\nabla^2 + 1)\psi\right]^2 + \frac{1}{4} \psi^4\right\}$$
Motivation
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- Mode amplitude $\psi_q(t)$ at wave vector $\mathbf{q}$ satisfies linear equation

$$\dot{\psi}_q = \tau_0^{-1} [\varepsilon - \xi_0^2 (q - q_c)^2] \psi_q$$
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- To be able to write this as a local equation for the Fourier transform $\psi(x, y, t)$ approximate this by

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• Add simplest possible nonlinear saturating term

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- Alternatively can think

$$A(x, y)e^{i_q x} \Rightarrow \psi(x, y)$$
Relaxation to steady state

(from Greenside and Coughran, 1984)
Coarsening in a periodic geometry

(From Elder, Vinals, and Grant 1992)
Generalized Swift-Hohenberg models

Qualitatively include other physics:
Generalized Swift-Hohenberg models

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• break $\psi \rightarrow -\psi$ symmetry

$$\partial_t \psi = \left[ r - (\nabla_\perp^2 + 1)^2 \right] \psi + \gamma \psi^2 - \psi^3$$
Generalized Swift-Hohenberg models

Qualitatively include other physics:

- **break** $\psi \rightarrow -\psi$ symmetry

\[ \partial_t \psi = \left[ r - (\nabla^2 + 1)^2 \right] \psi + \gamma \psi^2 - \psi^3 \]

- **add mean flow** $V$

\[(\partial_t + V \cdot \nabla)\psi = \left[ r - (\nabla^2 + 1)^2 \right] \psi - \psi^3 \]

$\nabla^2 V = g\hat{z} \cdot \nabla(\nabla^2 \psi) \times \nabla \psi$
Generalized Swift-Hohenberg models

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- change nonlinearity to make equation non-potential, e.g.

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$$\partial_t \psi = \left[ r - (\nabla^2_{\perp} + 1)^2 \right] \psi + (\nabla \psi)^2 \nabla^2 \psi$$

• model effects of rotation

$$\partial_t \psi = \left[ r - (\nabla^2_{\perp} + 1)^2 \right] \psi - \psi^3 + g_2 \hat{z} \cdot \nabla \times [ (\nabla \psi)^2 \nabla \psi ] + g_3 \nabla \cdot [ (\nabla \psi)^2 \nabla \psi ]$$
Phase dynamics
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• The phase variable describes the symmetry properties of the system: the connection between symmetry and slow dynamics is known as Goldstone’s theorem.

• Near threshold $\theta$ is simply the phase of the complex amplitude, and an equation for the phase dynamics can be derived from the amplitude equation for $\eta \ll \varepsilon$ (Pomeau and Manneville, 1979)
Equation for small phase distortions near threshold

For a phase variation \( \theta = kx + \delta \theta \)

\[
\partial_t \delta \theta = D_\parallel \partial_x^2 \delta \theta + D_\perp \partial_y^2 \delta \theta
\]

with diffusion constants for the state with wave number \( q = q_c + k \)

\[
D_\parallel = (\xi_0^2 \tau_0^{-1}) \frac{\epsilon - 3\xi_0^2 k^2}{\epsilon - \xi_0^2 k^2}
\]

\[
D_\perp = (\xi_0^2 \tau_0^{-1}) \frac{k}{q_c}.
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\]

A negative diffusion constant leads to exponentially growing solutions, i.e. the state with wave number \( q_c + k \) is unstable to long wavelength phase perturbations for

\[
|\xi_0 k| > \varepsilon^{1/2} / \sqrt{3} \quad \text{longitudinal (Eckhaus)}
\]

\[
k < 0 \quad \text{transverse (ZigZag)}
\]
Stability balloon near threshold

\[ \varepsilon \]

existence band

stable band

\[ q_N \]

\[ q-q_c \]
Phase dynamics away from threshold (MCC and Newell, 1984)

Away from threshold the other degrees of freedom relax even more quickly, and so idea of a slow phase equation remains.

- pattern is given by the lines of constant phase \( \theta \) of a local stripe solution;
- wave vector \( \mathbf{q} \) is the gradient of this phase \( \mathbf{q} = \nabla \theta \).
A nonlinear saturated straight-stripe solution with wave vector $q = q \hat{x}$ is

$$u = u_q(\theta, z, t) \quad \theta = qx$$
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For slow spatial variations of the wave vector over a length scale $\eta^{-1}$ this leads to the ansatz for a pattern of slowly varying stripes

$$u \approx u_q(\theta, z, t) + O(\eta), \quad q = \nabla \theta(x)$$

where $q = q(\eta x)$ so that $\nabla q = O(\eta)$. 
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where $\mathbf{q} = \mathbf{q}(\eta \mathbf{x})$ so that $\nabla \mathbf{q} = O(\eta)$.

We can develop an equation for the phase variation by expanding in $\eta$

$$
\tau(q) \partial_t \theta = -\nabla \cdot [\mathbf{q} B(q)]
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\]

The form of the equation derives from symmetry and smoothness arguments, and expanding up to second order derivatives of the phase.

The parameters \( \tau(q), B(q) \) are system dependent functions depending on the equations of motion, \( \mathbf{u}_q \), etc.
Small deviations from stripes

\[ \tau(q) \partial_r \theta = -\nabla \cdot [qB(q)] \]

For \( \theta = qx + \delta \theta \) this reduces to

\[ \partial_r \delta \theta = D_{||}(q) \partial_x^2 \delta \theta + D_{\perp}(q) \partial_y^2 \delta \theta \]

with

\[ D_{\perp}(q) = -\frac{B(q)}{\tau(q)} \]
\[ D_{||}(q) = -\frac{1}{\tau(q)} \frac{d(qB(q))}{dq} \]
Small deviations from stripes

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\[ D_{\perp}(q) = -\frac{B(q)}{\tau(q)} \]
\[ D_{\parallel}(q) = -\frac{1}{\tau(q)} \frac{d(q B(q))}{dq} \]

A negative diffusion constant signals instability:

- \([q B(q)]' < 0\): Eckhaus instability
- \(B(q) < 0\): zigzag instability
Phase parameters for the Swift-Hohenberg equation
Application: wave number selection by a focus

\[ \nabla \cdot (q B(q)) = 0 \Rightarrow \int B(q) q \cdot \hat{n} \, dl = 0 \]

\[ q B(q) = \frac{C}{r} \rightarrow 0 \quad r \to \infty \]

i.e. \( q \to q_f \) with \( B(q_f) = 0 \), the wave number of the zigzag instability!
Defects
Focus/target defect

Wavevector winding number = 1
Disclinations

Winding numbers: (a) $\frac{1}{2}$; (b) 1; (c) -1
Dislocation

\[
\text{Phase winding number } = \frac{1}{2\pi} \oint \nabla \theta \cdot dl = 1
\]
Dislocation climb

Smooth motion through symmetry related states

\[ v_d \approx \beta (q - q_d) \]
Dislocation glide

Motion involves stripe pinch off, and is pinned to the periodic structure
Spiral Dynamics: experiments of Plapp et al. (1998)
Dislocation motion

\[ v_d = \omega r_d = \beta (q(r_d) - q_d) \]  

Spiral motion from phase equation

\[ \tau_q \partial_r \theta = -\nabla \cdot [qB(q)] \]

\[ \omega = -\tau_q^{-1} \frac{1}{r} \frac{\partial}{\partial r} (rqB(q)) \]
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\[ q(r) - q_f = -\omega r/2\alpha + Cr^{-1}. \]

Evaluating at \( r_d \) and combining with Eq. (\ast) gives \( \omega \).
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Evaluating at \( r_d \) and combining with Eq. (\ast) gives \( \omega \).

Is this relevant to spiral defect chaos?
Lattice States

Stripe state

$q_x$
Square state
Rectangular (orthorhombic) state
Hexagonal state
Supersquare state
Superhexagon state
Quasicrystal state

\[ q_c \]
Amplitude equation description

Introduce amplitudes $A_i$ for each “component” set of stripes

$$\delta u(x_\perp, z, t) \approx \sum_i A_i(x_\perp, t) \times \left[ u_{q_c}q_i(z) e^{iq_c \hat{q}_i \cdot x_\perp} \right] + c.c.$$  

For no space dependence

$$\tau_0 \partial_t A_i = \varepsilon A_i - g_0 \left[ |A_i|^2 + \sum_{j \neq i} G(\theta_{ij}) |A_j|^2 \right] A_i$$

e.g. for squares would have $A_1 = A_2$ and $\theta_{12} = \pi/2$ so need to know $G(\pi/2)$.

Find stationary solutions and test for stability.
Hexagons without $u \rightarrow -u$ symmetry

Special case because $q_1 + q_2 + q_3 = 0$ leading to “3 mode resonance” terms

$$\tau_0 \partial_t A_1 = \varepsilon A_1 + \gamma A_2^* A_3^* - g_0 \left[ |A_1|^2 + \sum_{j \neq i} G(\pi/3) (|A_2|^2 + |A_3|^2) \right] A_1$$
Conclusions

In the second lecture I have described the implications of symmetry on the theoretical methods for stationary patterns:

- amplitude equation in 2d
- Swift-Hohenberg equation and generalizations
- phase equation

The methods have various advantages and disadvantages, and have given great insights, but none is a complete approach even near threshold.
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- topological defects
- competition between different planforms (stripes, lattices, quasicrystals).
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Next lecture: oscillatory instabilities.