Ph 12 Homework 7

1 Question 1

\[ \mathcal{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - V_L \delta(x + a) - V_R \delta(x - a) \]

1.1 Part a

In this section, we write down the solution for two bound modes assuming in turn that \( V_1 \) or \( V_2 \) is 0.

\[ |L\rangle = \begin{cases} \sqrt{\kappa_L} e^{\kappa_L(x+a)} & x \leq -a \\ -\sqrt{\kappa_L} e^{-\kappa_L(x+a)} & x > -a \end{cases} \]

\[ |R\rangle = \begin{cases} \sqrt{\kappa_R} e^{\kappa_R(x-a)} & x \leq a \\ -\sqrt{\kappa_R} e^{-\kappa_R(x-a)} & x > a \end{cases} \]

Aside: There are a number of ways of solving this problem - taking a variational approach to superpositions of isolated wave functions is probably the quickest and easiest.

Their decay constants are \( \kappa_L = \frac{m|V_L|}{\hbar^2} \) and \( \kappa_R = \frac{m|V_R|}{\hbar^2} \). Their energies are

\[ E_L = -\frac{mV_L^2}{2\hbar^2} \quad \text{and} \quad E_R = -\frac{mV_R^2}{2\hbar^2}. \]

1.2 Part b

As this problem lacks (in general) left-right symmetry, we cannot simply posit the existence of two combined states \(|+\rangle\) and \(|-\rangle\) as in the symmetrical case. Rather, in part (c) we shall see how they arise naturally as eigenvectors of the Hamiltonian. But now we have to calculate the Hamiltonian. In general, this is rather hard, but assuming \( \langle L|R \rangle \approx 0 \) will simplify matters considerably!

Consider a general superposition of \(|L\rangle\) and \(|R\rangle\), \( |\rangle = A|L\rangle + B|R\rangle \). Then we want to find a matrix \( \hat{H} \) such that \( \langle \mathcal{H} \rangle = \langle \hat{H} \rangle \), so we can switch from bra-ket notation to the much more intuitive matrix notation. The process is a bit bashy.

\[ \langle \mathcal{H} \rangle = (A\langle L| + B\langle R| \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - V_L \delta(x + a) - V_R \delta(x - a) \right) (A|L\rangle + B|R\rangle), \]

\[ = \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} \langle L| \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - V_L \delta(x + a) - V_R \delta(x - a) \right) |L\rangle \\ \langle L| \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - V_L \delta(x + a) - V_R \delta(x - a) \right) |R\rangle \\ \langle R| \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - V_L \delta(x + a) - V_R \delta(x - a) \right) |L\rangle \\ \langle R| \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - V_L \delta(x + a) - V_R \delta(x - a) \right) |R\rangle \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}. \]
Note that at this point, it is possible to show that the two off diagonal elements are Hermitian conjugates of each other. Given that all the fields and the Hamiltonian is real valued, this implies that the off diagonal elements are equal.

Proceeding rapidly, we exploit the original derivation and \( \langle L|R \rangle \approx 0 \) to write

\[
\hat{H} = \begin{pmatrix}
E_L - V_R(L)\delta(x-a)|L\rangle & -V_L(L)\delta(x+a)|R\rangle \\
-V_L(R)\delta(x+a)|L\rangle & V_R(L)\delta(x-a)|R\rangle
\end{pmatrix}.
\]

Bras and kets are shorthand for integrations over real space, and what could be simpler than an integral with a delta-function? As an example, \( \langle L|\delta(x-a)|L\rangle = \psi^*_L(a)\psi_L(a) = \kappa_Le^{-4\kappa_la} \). So

\[
\hat{H} = \begin{pmatrix}
E_L - V_R\kappa_Le^{-4\kappa_la} & -V_L\kappa_R\kappa_Re^{-2\kappa_{Ra}} - V_R\kappa_L\kappa_Re^{-2\kappa_{La}} \\
-V_L\kappa_L\kappa_Re^{-2\kappa_{La}} - V_R\kappa_R\kappa_Re^{-2\kappa_{Ra}} & E_R - V_L\kappa_R\kappa_Re^{-4\kappa_{Ra}}
\end{pmatrix}.
\]

Here we wave our hands a little bit, and say that \( e^{-2\kappa_a} \ll 1 \), which is true if our starting approximation is still valid. Then the square of this in the diagonal terms vanishes. Also, the off diagonal terms are equal and constant (in time), so we set \( J = -V_L\kappa_R\kappa_Re^{-2\kappa_{Ra}} - V_R\kappa_L\kappa_Re^{-2\kappa_{La}} \) and are left with

\[
\hat{H} = \begin{pmatrix}
E_L & J \\
J & E_R
\end{pmatrix}.
\]

In contrast to the Hamiltonian matrix for equal potentials (to which this reduces in the limit) the diagonal terms differ. The off diagonal term \( J \) is interpreted as the energy leaking from one mode to the other. If there were two different \( J \) terms, energy would pile up in one mode, which is not physical in this situation. Finally, this sort of matrix is Hermitian, and when exponentiated (with a factor of \( i \)) becomes unitary. The physical significance of this is that unitary matrices parameterise unitary transformations - think of rotations in the plane. These preserve geometric parameters of the system - in this case, the energy. The time evolution of a single wave function is written \( e^{-it\hat{H}/\hbar} \). For a quantum system with a non-trivial basis, this generalises to \( e^{-it\hat{H}/\hbar} \).

Note: using the tricks described in class, the Hamiltonian matrix could have been effectively written down in one line. One way to see that is that in the instance that \( B = 0 \) (no \( |R\rangle \)), the Hamiltonian has to return \( E_L \), and similarly for \( E_R \). Some degree of mode mixing \( J \) can be hypothesized by analogy with the symmetrical case, and added in. A more detailed calculation is only necessary if you want to find an explicit form for \( J \).

1.3 Part c

Don’t forget that \( E \) and \( J \) are negative by definition. By mathematica or inspection,

\[
E_\pm = \frac{1}{2} \left( E_L + E_R \pm \sqrt{(E_L - E_R)^2 + 4J^2} \right),
\]

so \( 0 > E_+ > E_- \) and (un-normalised) eigenvectors are

\[
|+\rangle = (E_+ - E_R, J)^T = (E_- - E_L, -J)^T,
\]
\[ |-\rangle = (E_- - E_R, J)^T = (E_+ - E_L, -J)^T. \]

### 1.4 Part d

There are a few ways of solving this. One such method is simply to exponentiate \( \hat{H} \) and find the answer, i.e.

\[
\begin{pmatrix} L \\ R \end{pmatrix} = e^{-i \begin{pmatrix} E_L & J \\ J & E_R \end{pmatrix} t} \begin{pmatrix} L_0 \\ R_0 \end{pmatrix}.
\]

This is mathematically correct, but physically void. These are not the natural basis sets from which to work in this system. Instead, use the eigenvectors and eigenvalues to diagonalise the Hamiltonian, thus greatly simplifying the problem. Setting \( \hbar = 1 \) the result is

\[
\begin{pmatrix} + \\ - \end{pmatrix} = \begin{pmatrix} \exp(-iE_+t) & 0 \\ 0 & \exp(-iE_-t) \end{pmatrix} \begin{pmatrix} +_0 \\ -_0 \end{pmatrix} = \begin{pmatrix} +_0 \exp(-iE_+t) \\ -_0 \exp(-iE_-t) \end{pmatrix}
\]

A bit of matrix kung-fu is now necessary. We have a formula which can evolve the diagonalised basis functions in time. So we express the initial state in terms of the L and R states, then time evolve them as + and - states, then re-express them as L and R states. First, let

\[
\Omega = \begin{pmatrix} E_+ - E_R & J \\ E_- - E_R & J \end{pmatrix}
\]

Then, for example, \( \hat{H}^\pm = \Omega \hat{H}_{LR} \Omega^{-1} \). Also, let \( LR \) and \( \pm \) represent respective ket states.

\[
|LR\rangle = \exp(-i\hat{H}_{LR}t)|LR_0\rangle,
\]

\[
= \Omega^{-1} \Omega \exp(-i\hat{H}_{LR}t) \Omega^{-1} \Omega |LR_0\rangle,
\]

\[
= \Omega^{-1} \exp(-i\hat{H}_\pm t) \Omega |LR_0\rangle.
\]

Logically, starting from the right, this equation converts the LR state into a \( \pm \) state, adds some phase (with the time evolution part), then converts it back again! Looking good. In full form:

\[
\begin{pmatrix} L \\ R \end{pmatrix} = \frac{1}{J(E_+ - E_-)} \begin{pmatrix} E_+ - E_R & -J \\ -E_- + E_R & E_+ - E_R \end{pmatrix} \begin{pmatrix} \exp(-iE_+t) & 0 \\ 0 & \exp(-iE_-t) \end{pmatrix} \begin{pmatrix} E_+ - E_R & J \\ E_- - E_R & J \end{pmatrix} \begin{pmatrix} L_0 \\ R_0 \end{pmatrix}
\]

Now you’ll really want to use mathematica!
From here one small step is needed to completion - multiplying through by initial amplitudes!

\[
\begin{pmatrix}
L \\
R
\end{pmatrix} = \frac{1}{J(E_- - E_+)} \times \\
\begin{pmatrix}
\frac{e^{-iE_-t}(E_- - E_R)J - e^{-iE_+t}(E_+ - E_R)J}{e^{-iE_-t} - e^{-iE_+t}} \\
-(e^{-iE_-t} - e^{-iE_+t})(E_- - E_R)(E_+ - E_R) - e^{-iE_-t}(E_+ - E_R)J + e^{-iE_+t}(E_- - E_R)J
\end{pmatrix} \begin{pmatrix}
L_0 \\
R_0
\end{pmatrix}.
\]

From here one small step is needed to completion - multiplying through by initial amplitudes!

\[
\begin{pmatrix}
L \\
R
\end{pmatrix} = \frac{1}{J(E_- - E_+)} \times \\
\begin{pmatrix}
J e^{-iE_-t}(E_-L_0 - E_RL_0 + JR_0) - Je^{-iE_+t}(E_+ - E_R + JR_0) \\
e^{-iE_-t}(E_+ - E_R)(-E_-L_0 + E_RL_0 - JR_0) + e^{-iE_+t}(E_- - E_R)(E_+L_0 - E_RL_0 + JR_0)
\end{pmatrix}.
\]

Note here that everything is expressed in terms of various energies \((E_R, E_L, E_+, E_-)\), which are in turn defined in terms of \(V_L\) and \(V_R\) and a few other terms that float around. However, in terms of answering the question, all that remains is to set \(L_0 = 1\) and \(R_0 = 0\). This gives (with \(h = 1\))

\[
\begin{pmatrix}
L \\
R
\end{pmatrix} = \frac{1}{J(E_- - E_+)} \begin{pmatrix}
J e^{-iE_-t}(E_- - E_R) - J e^{-iE_+t}(E_+ - E_R) \\
e^{-iE_-t} - e^{-iE_+t})(E_- - E_R)(E_+ - E_R)
\end{pmatrix}.
\]

Write \(E_{\pm} = E_{AV} \pm E_D\) as a simplified version of the quadratic formula we found earlier. Then

\[
\begin{pmatrix}
L \\
R
\end{pmatrix} = \frac{e^{-iE_{AV}t}}{JE_D} \begin{pmatrix}
JE_D\cos(E_Dt) - iJ(E_{AV} - E_R)\sin(E_Dt) \\
-i(E_D^2 - (E_{AV} - E_R)^2)\sin(E_Dt)
\end{pmatrix} = \begin{pmatrix}
a(t) \\
b(t)
\end{pmatrix}.
\]

For a basic consistency test, set \(t = 0\), and we see the state return to the initial condition \((1\ 0)^T\). A quick discussion is in order. As might be expected, the global phase crunches forward in time with a frequency of \(E_{AV} = (E_R + E_L)/2\), but otherwise has no effect on the amplitudes or probabilities of either state. Rabi oscillation is seen between the main brackets. Rabi oscillation occurs at a frequency of \(E_D\) in amplitude - observable probability density has twice the frequency. For the \(|L\rangle\) state, the dominant amplitude is driven by a cosine function, though there is a second part that acts as a sine in phase with the \(|R\rangle\) state. The states are coupled and share energy over a time period much longer than the fundamental frequency of either mode. That is, energy (which is proportional to the probability density) sloshes between the two bound modes with a frequency of \(2E_D \ll E_{AV}\). The smaller the splitting in energy levels between the two modes, the slower the oscillation.

Finally, the total probability density of the system is given by \(L^2 + R^2 = (L, R)(L, R)^\dagger\). The curious student can evaluate this expression or harmlessly speculate - what should it be?
2 Question 2

\[ \hat{L} = -i\hbar \mathbf{r} \times \nabla. \]

There are two ways of solving this problem. One is to perform the calculation in spherical coordinates, then convert the operator to cartesian coordinates, while keeping the derivatives of spherical coordinates. The other is to perform the calculation in cartesian coordinates, then use the Jacobian to convert cartesian derivatives into spherical polar derivatives. While both are mathematically equivalent, I shall be using the second approach in this solution.

2.1 Parts a, b, and c

For the time being, neglect the factor of \(-i\hbar\), as we will be concerned primarily with large matrices. As a matter of convention, I will be writing vectors as ordered triples in the orders \((x, y, z)\) and \((r, \theta, \phi)\).

\[ \hat{L} = \mathbf{r} \times \nabla, \]

\[
\hat{L} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\partial_x & \partial_y & \partial_z \\
x & y & z
\end{vmatrix},
\]

\[ = (y\partial_z - z\partial_y, z\partial_x - x\partial_z, x\partial_y - y\partial_z), \]

\[ = \begin{pmatrix}
0 & -z & y \\
z & 0 & -x \\
-y & x & 0
\end{pmatrix}
\begin{pmatrix}
\partial_x \\
\partial_y \\
\partial_z
\end{pmatrix}.\]

All cross products can be written in terms of an anti-symmetric matrix. Now we employ the Jacobian to convert the cartesian derivatives into spherical polar derivatives.

Jacobian refresher. In general, partial derivatives can be interchanged as follows.

\[
\frac{\partial}{\partial x} = \frac{\partial r}{\partial x} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x} \frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x} \frac{\partial}{\partial \phi}.\]

This generalises to

\[
\begin{pmatrix}
\partial_x \\
\partial_y \\
\partial_z
\end{pmatrix} = J
\begin{pmatrix}
\partial_r \\
\partial_\theta \\
\partial_\phi
\end{pmatrix},
\]

where

\[
J = \frac{\partial(r, \theta, \phi)}{\partial(x, y, z)} = \begin{pmatrix}
\frac{\partial r}{\partial x} & \frac{\partial \theta}{\partial x} & \frac{\partial \phi}{\partial x} \\
\frac{\partial r}{\partial y} & \frac{\partial \theta}{\partial y} & \frac{\partial \phi}{\partial y} \\
\frac{\partial r}{\partial z} & \frac{\partial \theta}{\partial z} & \frac{\partial \phi}{\partial z}
\end{pmatrix} = \begin{pmatrix}
x/r & y/r & -z/r \\
\sqrt{x^2 + y^2} & yz/r & 0 \\
\sqrt{x^2 + y^2} & x/r & 0
\end{pmatrix} = \begin{pmatrix}
x/r & y/r & -z/r \\
\sqrt{x^2 + y^2} & yz/r & 0 \\
\sqrt{x^2 + y^2} & x/r & 0
\end{pmatrix}.
\]
Source (though it is not difficult to calculate):
http://en.wikipedia.org/wiki/List_of_common_coordinate_transformations#From_Cartesian_coordinates

Note that I have abused notation slightly by writing \( r^2 = x^2 + y^2 + z^2 \) as shorthand. So

\[
\hat{L} = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \begin{pmatrix} \frac{x}{r} \\ \frac{y}{r} \\ \frac{z}{r} \end{pmatrix} \begin{pmatrix} \frac{r^2}{\sqrt{x^2+y^2}} & \frac{x}{x^2+y^2} & \frac{-y}{x^2+y^2} \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{y^2}{x^2+y^2} & \frac{-x}{x^2+y^2} \\ \frac{z}{\sqrt{x^2+y^2}} & \frac{z^2}{x^2+y^2} & 0 \end{pmatrix} \begin{pmatrix} \partial_r \\ \partial_\theta \\ \partial_\phi \end{pmatrix}
\]

\[
\begin{pmatrix} \partial_r \\ \partial_\theta \\ \partial_\phi \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{-y}{\sqrt{x^2+y^2}} \frac{\partial_\theta}{\partial_\phi} - \frac{z}{x^2+y^2} \frac{\partial_\phi}{\partial_\phi} \\ \frac{x}{\sqrt{x^2+y^2}} \frac{\partial_\theta}{\partial_\phi} - \frac{y}{x^2+y^2} \frac{\partial_\phi}{\partial_\phi} \end{pmatrix}.
\]

We have solved part a, b, and c in one fell swoop. That is, (returning \(-i\hbar\) to the equation),

\[
\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = -i\hbar \begin{pmatrix} \frac{-y}{\sqrt{x^2+y^2}} \frac{\partial_\theta}{\partial_\phi} - \frac{z}{x^2+y^2} \frac{\partial_\phi}{\partial_\phi} \\ \frac{x}{\sqrt{x^2+y^2}} \frac{\partial_\theta}{\partial_\phi} - \frac{y}{x^2+y^2} \frac{\partial_\phi}{\partial_\phi} \\ \frac{z}{\sqrt{x^2+y^2}} \frac{\partial_\theta}{\partial_\phi} - \frac{x}{x^2+y^2} \frac{\partial_\phi}{\partial_\phi} \end{pmatrix} = -i\hbar \begin{pmatrix} -\sin\phi \partial_\theta - \cos\phi\cot\theta \partial_\phi \\ \cos\phi \partial_\theta - \sin\phi\cot\theta \partial_\phi \end{pmatrix}.
\]

2.2 Part d

There are two ways to prove this. The more straight forward way is to return to the definition of \( \hat{L} \) in Cartesian coordinates with Cartesian derivatives, plug in the numbers, and every term in the commutator cancels except for the ones where a partial acts on the corresponding coordinate, returning the desired answer. The less straight forward way is to perform the operations on the Cartesian angular momentum operators in spherical polar coordinates and derivatives, though arguably the algebra is equivalently painful. As we shall see, every term which is a double derivative cancels.

\[
[L_z, L_x] = L_z L_x - L_x L_z
= -\hbar^2 \partial_\phi (-\sin\phi \partial_\theta - \cos\phi\cot\theta \partial_\phi) + \hbar^2 (-\sin\phi \partial_\theta - \cos\phi\cot\theta \partial_\phi) \partial_\phi
= -\hbar^2 (-\cos\phi \partial_\theta + \sin\phi\cot\theta \partial_\phi + (4 \text{ terms that cancel}))
= i\hbar L_y.
\]

As required. It is not difficult to show (easier in the Cartesian case) that all three commutation relations hold.
3 Question 3

At the time of writing, there was some confusion amongst students as to how this question works. Rather than dive into pages of arithmetic, an approach that focuses on intuition will be presented.

3.1 Part a

In this question, there are 3 identical, evenly spaced wells, which we assume are similar to the ones in Question 1. However, no specifics are given, and none are needed. We seek to construct a Hamiltonian matrix like the one in Question 1. This time, however, all the wells are equal. Since there are three states, we are looking for a $3 \times 3$ matrix. If the state of the system is $(1, 0, 0)$, then we expect the Hamiltonian to return the eigenenergy $\epsilon$. That is,

$$\epsilon = (1, 0, 0) \mathbf{H} (1, 0, 0)^T.$$  

It is pretty clear that the $(1, 1)$ entry of $\mathbf{H}$ must be $\epsilon$. By symmetry or similar arguments for the other states, we see that all the diagonal entries for $\mathbf{H}$ must be $\epsilon$. For the stoic, algebraically macho, mathematically inclined, or bored, this can be shown in a manner analogously to the method employed in Question 1, where the Hamiltonian includes a term for $3 \delta$-function potentials. As before, terms of order $(e^{-\kappa a})^2 = J^2 \approx 0$.

Now we must solve for the off diagonal elements. Remember that $J$ represents some overlap calculated like $\langle A|\delta(x-a)|B\rangle = A^*(a)B(a)$. Recall that for immediately adjacent wells, a term of order $J = e^{-\kappa a}$ appears, where $a$ is the distance between each well. For non-adjacent wells, this term then accounts for exponential decay over twice the distance, which is of order (or exactly) $J^2$, which we approximate to zero. Thus the para-diagonal terms in the Hamiltonian are all $J$, while the terms in the corners are 0. For systems with more wells, a general Hamiltonian is similarly tridiagonal to first order, which makes calculations a lot easier. For consistency with Question 1, I am defining $J < 0$ like $\epsilon$.

Thus

$$\mathbf{H} = \begin{pmatrix} \epsilon & J & 0 \\ J & \epsilon & J \\ 0 & J & \epsilon \end{pmatrix}.$$  

3.2 Part b

From here the solving process is reasonably straight forward. Use your favourite linear algebra package to diagonalise the system and find eigenvalues and eigenvectors. It is worth noting that due to symmetry we expect a STRONG correspondence with coupled harmonic oscillators as seen in term 1. Define $\Omega$ as the
matrix of eigenstates of this system (listed vertically).

\[
\Omega = \begin{pmatrix}
1 & \sqrt{2} & 1 \\
-1 & 0 & 1 \\
1 & -\sqrt{2} & 1
\end{pmatrix}.
\]

Then the diagonalised Hamiltonian is given by

\[
H_D = \Omega H \Omega^{-1} = \begin{pmatrix}
\epsilon + \sqrt{2}J & 0 & 0 \\
0 & \epsilon & 0 \\
0 & 0 & \epsilon - \sqrt{2}J
\end{pmatrix}.
\]

The diagonal entries are the eigenvalues (here eigenenergies), and the bottom one has the highest energy, because \(E\) and \(J\) are both negative. Each energy corresponds to the eigenstate shown in \(\Omega\), which can be seen roughly in the number of sign changes in each basis. That is, higher energy states have a higher frequency, so more spatial crinkling.

### 3.3 Part c

Rabi oscillations. This is a problem that requires time evolution - which is all our favourite. In Question 1, \(\hbar = 1\), but here we’ll do it properly for variety! The (hopefully by now familiar) Schrödinger equation is (in matrix notation)

\[
i\hbar \frac{\partial}{\partial t} \psi = H \psi.
\]

The solution of this equation (giving time evolution) is

\[
\psi(t) = \exp\left(-\frac{iH}{\hbar} t\right) \psi_0.
\]

Again, exponentiating a non-diagonal matrix is MUCH less computationally intensive than a diagonalised one, and also misses out all the nice physics. Proceeding,

\[
\psi(t) = \Omega^{-1} \exp\left(-\frac{iH}{\hbar} t\right) \Omega^{-1} \psi_0
\]

\[
= \Omega^{-1} \exp\left(-\frac{iH_D}{\hbar} t\right) \Omega \psi_0
\]

\[
= \Omega^{-1} \begin{pmatrix}
\exp\left(-\frac{i}{\hbar}(\epsilon + \sqrt{2}J)t\right) & 0 & 0 \\
0 & \exp\left(-\frac{i}{\hbar}\epsilon t\right) & 0 \\
0 & 0 & \exp\left(-\frac{i}{\hbar}(\epsilon - \sqrt{2}J)t\right)
\end{pmatrix} \Omega \psi_0.
\]

Again, an initial basis set \(\psi_0\) is converted into the diagonalised basis by \(\Omega\), the respective phases crunched and spat out by the exponentiated diagonal
Hamiltonian matrix, and then converted back into the basis set of individual wells by $\Omega^{-1}$.

How does this evolve in time? Immediately we can factor out the $\exp\left(-\frac{i}{\hbar} \epsilon t\right)$ term, which rotates the global phase, and thus isn’t seen in Rabi oscillations. Set $\psi_0 = (A, B, C)$, presumably normalised. Set $\omega = \sqrt{2} J/\hbar$.

$$
\psi(0) = e^{-\frac{i}{\hbar} \epsilon t} \Omega^{-1} \begin{pmatrix} e^{-iwt} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{iwt} \end{pmatrix} \Omega \begin{pmatrix} A \\ B \\ C \end{pmatrix}
= e^{-\frac{i}{\hbar} \epsilon t} \begin{pmatrix} \cos^2 \frac{\omega t}{2} & -\frac{i}{\sqrt{2}} \sin \omega t & -\sin^2 \frac{\omega t}{2} \\ -\frac{i}{\sqrt{2}} \sin \omega t & \cos \omega t & -\frac{i}{\sqrt{2}} \sin \omega t \\ -\sin^2 \frac{\omega t}{2} & -\frac{i}{\sqrt{2}} \sin \omega t & \cos^2 \frac{\omega t}{2} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix}
= e^{-\frac{i}{\hbar} \epsilon t} \begin{pmatrix} A \cos^2 \frac{\omega t}{2} - B \frac{1}{\sqrt{2}} \sin \omega t - C \sin^2 \frac{\omega t}{2} \\ -A \frac{1}{\sqrt{2}} \sin \omega t + B \cos \omega t - C \frac{1}{\sqrt{2}} \sin \omega t \\ -A \sin^2 \frac{\omega t}{2} - B \frac{1}{\sqrt{2}} \sin \omega t + C \cos^2 \frac{\omega t}{2} \end{pmatrix}.
$$

Note that our reward for choosing equal wells of equal separation is an excessively friendly looking answer. Immediately by inspection we can determine that the parts of the wave function oscillate at a frequency of $\omega$, which doubles when looking at the probability density.

To answer the question, the initial state is $\psi_0 = (0, 1, 0)$. So

$$
\psi(t) = e^{-\frac{i}{\hbar} \epsilon t} \begin{pmatrix} -\frac{i}{\sqrt{2}} \sin \omega t \\ \cos \omega t \\ -\frac{i}{\sqrt{2}} \sin \omega t \end{pmatrix}.
$$

This is interesting. The wave function is composed of real and imaginary parts. To get to the bottom of it, we’ll have to find the probability density $\psi^\dagger \psi$.

$$
\psi^\dagger \psi(t) = \begin{pmatrix} \frac{1}{2} \sin^2 \omega t & \frac{1}{2} \sin^2 \omega t \\ \frac{1}{2} \sin^2 \omega t & \frac{1}{2} \cos^2 \omega t \end{pmatrix}.
$$

$\sin^2 \omega t$ and $\cos^2 \omega t$ can both written in terms of $\cos 2\omega t$ terms which are in perfect antiphase. This verifies the question’s proposal that the particle appears to flow from the central well out to the sides and back again - at a frequency of $2\omega = 2\sqrt{2} J/\hbar$.

4 Concluding remark

These answers are written in much more detail than is necessary to get full marks (presuming they’re all correct...). I included extra detail to compensate in part for the difficulty of some of the problems and a few typographical mistakes in the notes, and the short time between now and the final.