Ph12b - Final

Instructions:

• This exam must be taken in one sitting over the period of 5 hours from start to end. Please note on the top of your solution the starting and finishing time. Do not write anything after the time has passed - it will not be graded.

• In terms of allowed material, you may only use any personal notes and problem set solutions, course hand-outs (i.e., material on the website), and one book of your choice during the exam. Note the textbook you are using in your solution.

• If you need to make any assumptions due to lack of information in the text, specify clearly your assumptions.

• Your solution is due in the Lauritsen 269 return-box by Wednesday, March 20th, 5pm.

• Do not discuss the exam with your colleagues before they (and you!) handed in their solution.
1. The return of the ring. In the midterm, you were requested to solve for the eigenstates and eigenenergies of a free particle on a ring of circumference \( L \):

\[
\mathcal{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}
\]  

(1)

with \( \psi(0) = \psi(L) \) and a continuous derivative for \( 0 \leq x \leq L \). The solutions were \(|n\rangle = \frac{1}{\sqrt{L}} e^{i k_n x} \) with \( k_n = 2\pi n/L \), and \( n \) an integer.

A particle is put in the ring in the state \(|\psi_0\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |2\rangle)\).

(a) (7) What is the probability density profile as a function of \( x \)?

(b) (5) The particle’s wave function evolves in time with the time-dependent Schrödinger equation, starting with \(|\psi_0\rangle\) as defined above (do not confuse it with \(|0\rangle\)!). What is the wave function as a function of time \( t \), \( \psi(x,t) \)?

(c) (7) What is the probability density as a function of \( x \) and \( t \)? What is the period (in time) of the motion described by this function?

(d) (6) What is the probability current along the ring as a function of \( x \) and \( t \)?

2. Axes of uncertainty. Since \( \hat{L}_z \) does not commute with \( \hat{L}_x \) or \( \hat{L}_y \), we know that eigenstates of \( \hat{L}_z \) are not eigenstates of \( \hat{L}_x \) and \( \hat{L}_y \). In this problem we will calculate the fluctuations and uncertainty of \( \hat{L}_y \) and \( \hat{L}_x \) for the \( \hat{L}_z \) and \( \hat{L}^2 \) eigenstates, \(|\ell, m\rangle\). Note that \( \hat{L}^2 \langle \ell, m | \ell, m \rangle = \hbar^2 (\ell + 1) |\ell, m \rangle \) and \( \hat{L}_z |\ell, m \rangle = \hbar m |\ell, m \rangle \).

(a) (8) By writing \( \hat{L}_x \) and \( \hat{L}_y \) in terms of the raising and lowering operators \( \hat{L}_+ \) and \( \hat{L}_- \), find the expectation values \( \langle \hat{L}_x \rangle \) and \( \langle \hat{L}_y \rangle \) for the state \(|\ell, m \rangle\).

(b) (9) What is the expectation value in the state \(|\ell, m \rangle\) of the operators \( \hat{L}_x^2 \) and \( \hat{L}_y^2 \)?

(c) (8) The uncertainty of an operator in a state \(|\psi\rangle\) was defined in class to be:

\[
\Delta A = \sqrt{\langle \psi | \hat{A}^2 | \psi \rangle - \langle \langle \psi | \hat{A} | \psi \rangle \rangle^2}.
\]  

(2)

Furthermore, we showed that for two operators \( \hat{A} \) and \( \hat{B} \) the Heisenberg uncertainty relation is:

\[
\Delta A \Delta B \geq \frac{1}{2} \langle [\hat{A}, \hat{B}] \rangle.
\]  

(3)

where all expectation values are with respect to the same \(|\psi\rangle\) state.

Find the appropriate lower limit on the uncertainty of the product \( \Delta L_x \Delta L_y \) in the state \(|\ell, m \rangle\). Show explicitly, using your answers, that the Heisenberg uncertainty relation is obeyed for all states \(|\ell, m \rangle\).

Not completely necessary, but Michael requested easy access to the following formulas:

\[
\hat{L}_+ |\ell, m \rangle = \hbar \sqrt{\ell (\ell + 1) - m (m + 1)} |\ell, m + 1 \rangle, \quad \hat{L}_- |\ell, m \rangle = \hbar \sqrt{\ell (\ell + 1) - m (m + 1)} |\ell, m - 1 \rangle
\]

3. Quantum mechanics in a nutshell. A particle of mass \( m \) is confined to a spherical shell centered at \( r = 0 \), and described as the potential well:

\[
V(r) = \begin{cases} 
\infty & r < a \\
0 & a < r < 2a \\
\infty & r > 2a
\end{cases}
\]  

(4)

(a) (8) What is the (time independent) Schrödinger equation describing this particle? Feel free to use the angular momentum operator \( \hat{L} \) to describe the angular part of the Hamiltonian.

(b) (7) Assuming the particle is in an angular momentum eigenstate, what is the radial Schrödinger equation for the particle?
(c) (10) Find the Hamiltonian’s eigenstates with zero-angular momentum. What are their energies? What are their wavefunctions?

4. A thorn in my side (or center). A particle of mass \( m \) is confined to a one-dimensional box, between \(-a < x < a\), with the walls of the box having an infinite positive potential. An attractive delta function \( V(x) = -aC\delta(x) \) is at the center of the box.

(a) (9) What are the energy eigenvalues of the particle of the odd-parity (i.e., those that are -1 times the wave function obtained by reflecting them about \( x=0 \)) eigenstates?

(b) (8) Find the value of \( C \) for which the lowest energy eigenvalue is zero. For this purpose, write out the Schrödinger equation with energy \( E = 0 \), and find its possible solutions for the regions \( x < 0 \) and \( x > 0 \).

(c) (8) When \( C \) is larger than the value found above, negative-energy solutions will exist. Given that the energy of the negative energy solution is \( E = -\epsilon \) with \( \epsilon > 0 \), what is the state’s wave function? Again rely on the Schrödinger equation at the regions \( 0 < |x| \leq a \).