Ph 12b Midterm Solutions

Problem 1

Our system is composed by a particle confined on a ring, and it is described by the hamiltonian

\[ H |\psi\rangle = E |\psi\rangle \]  

(1)

where \( H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \) is the selfadjoint operator that represents it.

a. We want to find eigenstates of this hamiltonian \( \{\psi_n(x)\} \) with the constraint that the particle’s wavefunction should be periodic and single-valued on the ring. The eigenstates of the hamiltonian above are the well-known plane waves \( \frac{1}{\sqrt{L}} e^{ikx} \) and the constraint we’ve just stated translates into \( e^{ikL} = 1 \), which gives

\[ kL = 2\pi n, \quad k_n = \frac{2\pi n}{L} \]  

(2)

so that we get for the spectrum of the particle

\[ E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{2\hbar^2 \pi^2 n^2}{mL^2} \]  

(3)

whereas the eigenstates are the wavefunctions \( \psi_n(x) = \sqrt{\frac{1}{L}} e^{ik_n x} \).

b. The Bohr Sommerfeld quantization condition requires that

\[ \int p \cdot dl = 2\pi \hbar n \]

so that we get in our case

\[ p_n = \frac{2\pi n \hbar}{L} \rightarrow k_n = \frac{2\pi n}{L} \]

and we recover the result found in the first point.

c. Now the particle is described at \( t = 0 \) by the wavefunction

\[ \psi(x,0) = 4\sqrt{\frac{8}{3L}} \sin^2\left(\frac{2\pi x}{L}\right) \cos^2\left(\frac{2\pi x}{L}\right). \]  

(4)

Using the usual trigonometric relations we can express this wavefunction into eigenstates of the hamiltonian:

\[ \psi(x,0) = \sqrt{\frac{8}{3L}} \sin^2\left(\frac{4\pi x}{L}\right) \rightarrow \sqrt{\frac{2}{3L}} \left(1 - \cos\left(\frac{8\pi x}{L}\right)\right) \]  

(5)
Now \( \cos\left(\frac{8\pi x}{L}\right) \), being a superposition of two eigenstates of \( H \) with the same energy, is itself an eigenstate of \( H \), with energy \( E_n = \frac{32\hbar^2}{mL^2} \), and so we get for the time evolution of our wavefunction:

\[
\psi(x, t) = \sqrt{\frac{2}{3L}} (1 - \cos\left(\frac{8\pi x}{L}\right)) e^{-i \frac{32\hbar^2}{mL^2} t}
\]  

(d) the wavefunction, after the collapse, is described by the eigenstate of \( H \) with zero momentum, which is the wavefunction \( \psi_{n=0}(x) \):

\[
\psi(x) = \sqrt{\frac{1}{L}}
\]

Problem 2

(a) When \( p \ll mc \) we can Taylor expand the expression for the energy

\[
E = mc^2 \sqrt{1 + \frac{p^2 c^2}{m^2 c^4}} = mc^2 \sqrt{1 + \frac{\hat{p}^2}{m^2 c^2}} = mc^2 + \frac{p^2}{2m} - \frac{1}{8} \frac{mc^2 \hat{p}^4 c^4}{m^4 c^4} = mc^2 + \frac{p^2}{2m} - \frac{p^4}{8m^3 c^2}
\]

(b) Hamiltonian:

\[
H = \frac{1}{2m} \hat{p}^2 - \frac{\hat{p}^4}{8m^3 c^2} = -\frac{1}{2m} \frac{\partial^2}{\partial^2 x} - \frac{\hbar^2}{8m^3 c^2} \frac{\partial^4}{\partial^4 x}
\]

Schrodinger equation in the momentum representation reads

\[
i\hbar \frac{\partial}{\partial \hat{p}} \Psi = \left( \frac{\hat{p}^2}{2m} - \frac{\hat{p}^4}{8m^3 c^4} \right) \Psi
\]

\[
\Psi(p, t) = \Psi(p, 0)e^{-i \frac{1}{\hbar} \left( \frac{p^2}{2m} - \frac{p^4}{8m^3 c^4} \right)t}
\]

(c) Let us set \( \hbar = 1 \).

\[
\Psi(k, t) = \sqrt{\frac{2\pi}{\Delta k}} e^{-ikx_0 - \frac{(k-k_0)^2}{4\Delta k^2}} e^{-i \frac{k^2}{8m^3 c^2} t}
\]

We see that the spectrum peaks at \( k_0 \).

The particle moves in real space with velocity:

\[
\langle v_{\text{group}} \rangle = \frac{\partial \omega}{\partial k} \bigg|_{k_0} = -\frac{\partial}{\partial k} \left( \frac{k^2}{2m} - \frac{k^4}{8m^3 c^2} \right) \bigg|_{k_0} = k_0 - \frac{k_0^3}{8m c^2}.
\]

Phase profile moves with velocity:

\[
\langle v_{\text{phase}} \rangle = \frac{\omega}{k} \bigg|_{k_0} = \frac{k^2}{2m} - \frac{k^4}{8m^3 c^2} \bigg|_{k_0} = k_0 - \frac{k_0^3}{8m c^2}.
\]

(d) Recall that \( p = \hbar k \) and \( E = \hbar \omega \).

\[
E = \sqrt{m^2 c^4 + \frac{p^2 c^2}{m^2 c^4}}
\]

\[
E = \sqrt{m^2 c^4 + \hbar^2 k^2 c^2}
\]

\[
\omega = \frac{E}{\hbar} = \sqrt{\frac{k^2 c^2}{\hbar^2} + \frac{m^2 c^4}{\hbar^2}}
\]

Group velocity:

\[
v_{\text{group}} = \frac{\partial \omega}{\partial k} = c \frac{k}{\sqrt{k^2 c^2 + \frac{m^2 c^4}{\hbar^2}}} < 0
\]

Phase velocity:

\[
v_{\text{phase}} = \frac{\omega}{k} = c \sqrt{1 + \frac{m^2 c^4}{\hbar^2 k^2}}
\]
Problem 3

a) From class and HW2 problem 3 e) we know that

\[ \bar{\psi}(k, t) = e^{-i\frac{\hbar k^2 t}{2m}} \psi(k, 0). \]

b) \[
\langle \hat{x} \rangle_t = \int dk \bar{\psi}^*(k, t) \hat{x} \bar{\psi}(k, t) = i \int dk \bar{\psi}^*(k, 0) e^{\frac{i\hbar k^2 t}{2m}} \left( e^{-i\frac{\hbar k^2 t}{2m}} \bar{\psi}(k, 0) \right),
\]

\[ \frac{t}{m} \int dk \bar{\psi}^*(k, 0) \hbar k \bar{\psi}(k, 0) + \int dk \bar{\psi}^*(k, 0) i \frac{\partial}{\partial k} \bar{\psi}(k, 0), \]

\[ = \langle \bar{x} \rangle_0 + \langle \bar{p} \rangle_0 \frac{t}{m}. \]

This correspond to the classical result for the motion of a free particle \( x(t) = x(0) + \bar{p} \cdot t. \)

c) \[
\langle \hat{x}^2 \rangle_t = \int dk \bar{\psi}^*(k, 0) e^{\frac{i\hbar k^2 t}{2m}} \left( i \frac{\partial}{\partial k} \right) \left( e^{-i\frac{\hbar k^2 t}{2m}} \bar{\psi}(k, 0) \right),
\]

\[ = \int dk \bar{\psi}^*(k, 0) e^{\frac{i\hbar k^2 t}{2m}} \left( \frac{\partial}{\partial k} \right) \left( \psi(k, 0) \right),
\]

\[ = \int dk \bar{\psi}^*(k, 0) \left( \frac{-i\hbar t}{m} \right) \left( \psi(k, 0) \right),
\]

\[ = \int dk \bar{\psi}^*(k, 0) \left( \frac{-i\hbar t}{m} \right) \left( \psi(k, 0) \right),
\]

\[ = \langle \hat{x}^2 \rangle_0 + \langle \hat{p} \hat{p} + \hat{p} \rangle_0 \frac{t^2}{m^2} \]

\[ = \frac{\langle \hat{x}^2 \rangle_0 + \langle \hat{p} \rangle_0}{m} \frac{t^2}{m^2}. \]

d) We know that \( \sigma_x(t) = \left( \langle \hat{x}^2 \rangle_t - \langle \hat{x} \rangle_t^2 \right)^{1/2} \) and from b) and c) we know that we can find \( \langle \hat{x}^2 \rangle_t \) and \( \langle \hat{x} \rangle_t \)

if we know the numerical values of the \( \langle \hat{x}^2 \rangle_0, \langle \hat{p}_x \rangle_0, \langle \hat{p}_y \rangle_0, \text{ and } \langle \hat{p}_x \hat{p}_y \rangle_0 \) for our wave function.

First, it is given in the problem that \( \langle \hat{p}_x + \hat{p}_y \rangle_0 = 0. \)

It is easy to see that \( \langle \hat{x} \rangle_0 = \langle \hat{p} \rangle_0 = 0 \) by symmetry, because \( \psi(x, 0) \) is even in this problem. This imply that \( \langle \hat{x} \rangle_t = 0. \)

Now, the given wave function is

\[ \psi(x, 0) = \left( \frac{\alpha}{\pi} \right)^{1/4} \exp \left( -\frac{\alpha x^2}{2} \right), \]

which is just a Gaussian distribution and it is well known that

\[ \langle \hat{x}^2 \rangle_0 = 1/2 \alpha. \]
An easy way to get this result is

\[ 1 = \int \psi^*(x, 0)\psi(x, 0)dx = \int \left(\frac{\alpha}{\pi}\right)^{1/2} \exp\left(-\alpha x^2\right), \]

by normalization, now we can take the derivative with respect to \( \alpha \) and we get

\[ 0 = \int \frac{1}{2} \left(\frac{\alpha}{\pi}\right)^{1/2} \exp\left(-\alpha x^2\right) - \int x^2 \left(\frac{\alpha}{\pi}\right)^{1/2} \exp\left(-\alpha x^2\right), \]

then \( \langle \hat{x}^2 \rangle_0 = 1/2\alpha. \)

Since we know that the Gaussian distribution has the minimum uncertainty (and \( \langle \hat{x} \rangle_0 = \langle \hat{p} \rangle_0 = 0 \)), we know that

\[ \langle \hat{x}^2 \rangle_0 \langle \hat{p}^2 \rangle_0 = \hbar^2/4, \]

then

\[ \langle \hat{p}^2 \rangle_0 = \frac{\hbar^2 \alpha}{2}. \]

Finally

\[ \sigma_x(t) = \sqrt{\frac{\hbar^2 \alpha}{2m^2} + \frac{1}{2\alpha}}. \]

**Problem 4**

(a) Schrödinger equation: \( i\hbar \frac{\partial}{\partial t} \Psi = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + \beta|x|\Psi \)

(b) Force acting on particle is \( F = -\frac{\partial V(x)}{\partial x} \).

For \( x \geq 0 \) we have the following equation of motion

\[
\begin{align*}
m\ddot{x}(t) &= -\beta \\
\dot{x}(t) &= -\frac{\beta}{m} \\
\dot{\dot{x}}(t) &= -\frac{\beta}{m} t + v_0 \\
x(t) &= -\frac{\beta}{2m} t^2 + v_0 t
\end{align*}
\]

For \( x < 0 \) we have \( x(t) = \frac{\beta}{2m} t^2 - v_0 t \).

The amplitude of oscillations \( x_{\text{max}} \) can be found as the point on the tragedtory with zero kinetic energy:

\[ E = \frac{mv_0^2}{2} = \beta x_{\text{max}}, \quad x_{\text{max}} = \frac{mv_0^2}{2\beta}. \]

(c) Let us calculate the momentum, which we then substitute into the action

\[ E = \frac{mv_0^2}{2} = \frac{p^2}{2m} + \beta|x| \quad \Rightarrow \quad \frac{p^2}{2m} = \frac{mv_0^2}{2} - \beta|x| \quad \Rightarrow \quad p = \sqrt{m^2 v_0^2 - 2m\beta|x|}. \]
The expression for the action reads

\[ S = 4 \int dx \sqrt{m^2 v_0^2 - 2m\beta |x|} = -\frac{4}{3m\beta} \left( m^2 v_0^2 - 2m\beta |x| \right)^{3/2} \bigg|_{x_0 = x_{\text{max}}} = \frac{4m^2 v_0^3}{3\beta}. \]

(d) Using Bohr-Sommerfield rule:

\[ \frac{4m^2 v_0^3}{3\beta} = n\hbar \quad \Rightarrow \quad v_0^3 = \frac{3\beta n\hbar}{4m^2} \quad \Rightarrow \quad v_0 = \left( \frac{3\beta n\hbar}{4m^2} \right)^{1/3}. \]  

(e) Total energy: \( E \sim v_0^2 \sim n^{2/3}. \)